Proceedings of the 7th GACM Colloquium on Computational Mechanics for Young Scientists from Academia and Industry
October 11-13, 2017 in Stuttgart, Germany

A robust explicit finite element algorithm with bipenalty stabilization for contact-impact problems

Radek Kolman[1]*, Ján Kopačka[1], Anton Tkachuk[2], Dušan Gabriel[1], José González[3] and Manfred Bischoff[2]

Micro Abstract
In this paper, an explicit time integration scheme for finite element solution of contact-impact problems with stabilization of contact forces is presented. The stability limit for an un-penalized system is preserved by the bipenalty method, i.e. a special choice of mass and stiffness penalty parameter ratio. Moreover, the time stepping process produces stable results for a large range of the stiffness penalty parameter. Behavior of the method is shown on a one-dimensional impact problem of elastic bars.

[1]Institute of Thermomechanics, Czech Academy of Sciences, Prague, Czech Republic
[2]Institute for Structural Mechanics, University of Stuttgart, Stuttgart, Germany
[3]Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Seville, Spain
***Corresponding author**: kolman@it.cas.cz

Introduction

Accurate and robust numerical modelling for contact-impact problems is still an up-to-date and open problem. Frequently, penalty methods, Lagrange Multiplier methods or augmented Lagrangian methods and more others can be applied for modelling of dynamic contact problems, e.g. see [2, 3, 10]. In explicit finite element method, the penalty method is preferred due to its simplicity [2]. On the other hand, in the penalty method, the stability limit is cardinally destroyed by a large value of numerical stiffness penalty parameter for enforcing contact constraints. One way how to eliminate this effect is to use the bipenalty method [5].

A big trouble in numerical modelling of contact-impact problems comes from spurious oscillations of contact forces often caused by activation and deactivation of contact constraints during computations of a response. There exist a lot of numerical techniques and strategies for elimination and stabilization of solution due to spurious oscillations on contact surfaces. We can mention the stabilized implicit Newmark method for non-smooth dynamics and contacts [4, 7], mass redistribution techniques [8], singular mass techniques [11, 12] or the stabilized explicit scheme with penalty method [14].

In this paper, we present an explicit time integration scheme for finite element solution of contact-impact problems with stabilization of contact forces based on work [14] in combination with the bipenalty formulation [9]. Superior behavior of the presented method for modelling of contact-impact problems is shown on an impact problem of elastic bars and commented. The obtained results are compared with a standard time integration scheme in explicit finite element method for impact-contact problems - the central difference method [1].

1 Bipenalty method in finite element method for contact-impact problems

In the finite element procedures [1] for elastodynamic problems with contact constraints, the equations of motion yield the following system of nonlinear ordinary differential equations

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{R}(t) - \mathbf{R}_{\mathrm{c}}(\mathbf{u}, \ddot{\mathbf{u}}) \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}$ and $\ddot{\mathbf{u}}$ are nodal displacements and accelerations, $\mathbf{R}$ is the vector of external loading with time dependency, t is the time, $\mathbf{R}_{\mathrm{c}}$ is the vector of contact forces. In the bipenalty formulation [9], the global contact residual vector $\mathbf{R}_{\mathrm{c}}$ is assembled from the local counterpart $\hat{\mathbf{R}}_{\mathrm{c}}$ as the contribution of stiffness and mass terms to contact residual vector which can be written as

$$\hat{\mathbf{R}}_{\mathrm{c}}(\hat{\mathbf{u}}, \ddot{\hat{\mathbf{u}}}) = \hat{\mathbf{M}}_{\mathrm{p}}\ddot{\hat{\mathbf{u}}} + \hat{\mathbf{K}}_{\mathrm{p}}\hat{\mathbf{u}} + \hat{\mathbf{f}}_{\mathrm{p}} \tag{2}$$

where

$$\hat{\mathbf{M}}_{\mathrm{p}} = \int_{\Gamma_{\mathrm{c}}} \epsilon_{\mathrm{m}} H(g)\mathbf{N}\mathbf{N}^{\mathrm{T}}\,\mathrm{d}S \qquad \hat{\mathbf{K}}_{\mathrm{p}} = \int_{\Gamma_{\mathrm{c}}} \epsilon_{\mathrm{s}} H(g)\mathbf{N}\mathbf{N}^{\mathrm{T}}\,\mathrm{d}S \qquad \hat{\mathbf{f}}_{\mathrm{p}} = \int_{\Gamma_{\mathrm{c}}} \epsilon_{\mathrm{s}} H(g)\mathbf{N}g_0\,\mathrm{d}S \tag{3}$$

Here, $\hat{\mathbf{M}}_{\mathrm{p}}$ is the additional elemental mass matrix due to inertia penalty, $\hat{\mathbf{K}}_{\mathrm{p}}$ is the additional elemental stiffness matrix due to stiffness penalty, and $\hat{\mathbf{f}}_{\mathrm{p}}$ is the part of the elemental contact force due to the initial gap g_0; g is the gap function; $H(g)$ is the Heaviside step function for prescribing active or inactive contact constraints; ϵ_{m} and ϵ_{s} are mass and stiffness penalty parameters; Γ_{c} is the contact surface between bodies; the matrix $\mathbf{N}$ represents an operator from the displacement field $\mathbf{u}$ to the gap function g_{N} in the contact

$$g_{\mathrm{N}} = \mathbf{N}^{\mathrm{T}}\mathbf{u} + g_0 \tag{4}$$

The particular form of the matrix $\mathbf{N}$ follows from the used contact discretization. A comprehensive overview can be found e.g. in [13].

2 Explicit time integration schemes for contact-impact problems

We now consider the time integration of the semi-discretized system (1) in the framework of the central difference method [1]

$$(\mathbf{M}^t + \mathbf{M}_{\mathrm{p}}^t)\frac{\mathbf{u}^{t+\Delta t} - 2\mathbf{u}^t + \mathbf{u}^{t-\Delta t}}{\Delta t^2} + (\mathbf{K}^t + \mathbf{K}_{\mathrm{p}}^t)\mathbf{u}^t + \mathbf{f}_{\mathrm{p}}^t - \mathbf{R}^t = \mathbf{0} \tag{5}$$

Assuming that displacements are known at time $t - \Delta t$ and t, one can resolve unknown displacements at time $t + \Delta t$, where Δt marks the time step size. Note, that the matrices $\mathbf{M}_{\mathrm{p}}^t$ and $\mathbf{K}_{\mathrm{p}}^t$ are time-dependent because they are associated with active contact constraints.

Explicit central difference scheme for contact-impact problems: In this paper, we use the following form of the central difference scheme for solving elastodynamic problems with contact constraints [1] with the flowchart:

- Given $\mathbf{u}^t$, $\dot{\mathbf{u}}^{t-\Delta t/2}$, $\mathbf{R}^t$
- For given $\mathbf{u}^t$ analyze contact, compute gap vector $\mathbf{g}$ and contact forces $\mathbf{f}_{\mathrm{p}}^t = -\mathbf{K}_{\mathrm{p}}^t\mathbf{u}^t + \mathbf{f}_{\mathrm{p}}^0$
- Compute accelerations $\ddot{\mathbf{u}}^t = (\mathbf{M}^t + \mathbf{M}_{\mathrm{p}}^t)^{-1}(\mathbf{R}^t - \mathbf{K}^t\mathbf{u}^t + \mathbf{f}_{\mathrm{p}}^t)$
- Mid-point velocities $\dot{\mathbf{u}}^{t+\Delta t/2} = \dot{\mathbf{u}}^{t-\Delta t/2} + \Delta t\ddot{\mathbf{u}}^t$
- New displacements $\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \Delta t\dot{\mathbf{u}}^{t+\Delta t/2}$
- $t \to t + \Delta t$

Here, we used the lumped version of mass matrix $\mathbf{M}$ by the row-summing.

Stabilized explicit time integration scheme for contact-impact problems: In the work of Wu [14], the fully explicit time integration scheme with stabilized technique for contact-impact problems has been published and tested. The mentioned time integration scheme takes the following flowchart with splitting of bulk and contact accelerations:

- Given $\mathbf{u}^t$, $\dot{\mathbf{u}}^{t-\Delta t/2}$, $\mathbf{R}^t$
- Compute accelerations of predictor phase $\ddot{\mathbf{u}}^t_{pred} = \mathbf{M}^{-1}(\mathbf{R}^t - \mathbf{K}\mathbf{u}^t)$
- Mid-point velocities of predictor phase $\dot{\mathbf{u}}^{t+\Delta t/2}_{pred} = \dot{\mathbf{u}}^{t-\Delta t/2} + \Delta t \ddot{\mathbf{u}}^t_{pred}$
- Displacements of predictor phase $\mathbf{u}^{t+\Delta t}_{pred} = \mathbf{u}^t + \Delta t \dot{\mathbf{u}}^{t+\Delta t/2}_{pred}$
- For given $\mathbf{u}^{t+\Delta t}_{pred}$ analyze contact, compute gap vector $\mathbf{g}$ and contact forces $\mathbf{f}_{\mathrm{p}\,pred} = -\mathbf{K}_{\mathrm{p}}\mathbf{u}^{t+\Delta t}_{pred} + \mathbf{f}^0_{\mathrm{p}}$
- Compute accelerations of corrector phase $\ddot{\mathbf{u}}^t_{corr} = (\mathbf{M} + \mathbf{M}_{\mathrm{p}})^{-1}(\mathbf{f}_{\mathrm{p}\,pred})$
- Compute total accelerations $\ddot{\mathbf{u}}^t = \ddot{\mathbf{u}}^t_{pred} + \ddot{\mathbf{u}}^t_{corr}$
- Mid-point velocities of corrector phase $\dot{\mathbf{u}}^{t+\Delta t/2} = \dot{\mathbf{u}}^{t+\Delta t/2}_{pred} + \Delta t \ddot{\mathbf{u}}^t_{corr}$
- New displacements of corrector phase $\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \Delta t \dot{\mathbf{u}}^{t+\Delta t/2}$
- For given $\mathbf{u}^{t+\Delta t}$ analyze contact, compute gap vector $\mathbf{g}$ and contact forces $\mathbf{f}^{t+\Delta t}_{\mathrm{p}} = -\mathbf{K}_{\mathrm{p}}\mathbf{u}^{t+\Delta t} + \mathbf{f}^0_{\mathrm{p}}$
- $t \rightarrow t + \Delta t$

In this two-time step scheme, bulk accelerations in the predictor phase $\ddot{\mathbf{u}}^t_{pred}$ are due to internal and external forces and they are computed with the standard lumped mass matrix as for a contact-free problem. After updating of velocities and displacements, contact constraints are analyzed and contact forces $\mathbf{f}_{\mathrm{p}\,pred}$ are computed. For these contact forces, contact accelerations in the corrector phase are computed with the additional penalized mass matrix.

Both mentioned explicit time integration schemes are tested in the numerical benchmark below.

3 Stability limit of the bipenalty method

It is known that the standard penalty method [2], where an additional stiffness term corresponding to contact boundary conditions is applied, significantly attacks the stability limit of the finite element model. Generally, the critical time step size rapidly decreases with increasing penalty stiffness [2]. On the other hand, this numerical effect can be eliminated by a special choice of additional mass penalty term - the bipenalty method [5]. The stability limit for the bipenalty method has been studied in [9], where the optimal ratio of stiffness and mass penalty parameters were found. The critical time step size associated to contact-free bodies is preserved for this optimal setting of mass penalty parameter with respect to the stiffness penalty parameter. Thus stability limit for the contact problem is not attacked by the stiffness penalty term. In principle, one can integrate contact-impact problems by an arbitrary stable time step size.

4 Numerical test - one-dimensional impact of two bars with different lengths

In this example, we study a one-dimensional contact-impact problem of two elastic bars with different lengths defined in [6]. A scheme of this test is depicted in Figure 1. The left bar is moving to the right with a constant velocity $v_{01} = 0.1\,[\mathrm{m/s}]$. The right bar with fixed right-hand side is at rest. The geometrical, material and numerical parameters were set up: the lengths $L_1 = 10\,[\mathrm{m}]$ and $L_2 = 20\,[\mathrm{m}]$, the Young's modulus $E_1 = E_2 = 100\,[\mathrm{Pa}]$, the mass density $\rho_1 = \rho_2 = 0.01\,[\mathrm{kg \cdot m^{-3}}]$, the cross-sectional area $A_1 = A_2 = 1\,[\mathrm{m^2}]$, the number of finite linear elements for each bar $n_1 = 50$, $n_2 = 100$, thus the finite element lengths are set up as $h_1 = h_2 = 0.2\,[\mathrm{m}]$, the initial contact gap $g_0 = 0\,[\mathrm{m}]$, the duration time $T = 0.7\,[\mathrm{s}]$. The value of the contact force from the analytical prediction is $R_{\mathrm{c}} = 0.05\,[\mathrm{N}]$ for $t = 0 \ldots 0.2\,[\mathrm{s}]$ and $t = 0.4 \ldots 0.6\,[\mathrm{s}]$ and zero otherwise. The standard lumped mass matrix and "consistent" mass penalty matrix were used in computations.

Let us define dimensionless stiffness and mass penalty parameters for one-dimensional cases, β_{s} and β_{m}, as follows

$$\beta_{\mathrm{s}} = \frac{h_e}{E}\epsilon_{\mathrm{s}}, \quad \beta_{\mathrm{m}} = \frac{2}{\rho h_e}\epsilon_{\mathrm{m}} \tag{6}$$

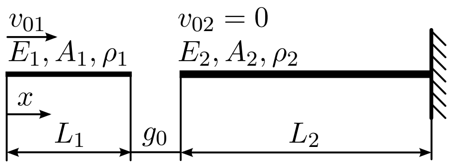

Figure 1. A scheme of an one-dimensional impact of two bars with different lengths.

where h_e is the length of the finite element whose master node is on the contact interface. In [9], the optimal penalty ratio of stiffness and mass penalty parameters for one-dimensinal cases is given by the value $\beta_{\mathrm{m}} = \frac{1}{2}\beta_{\mathrm{s}}$. The time step size was chosen as $\Delta t = 0.5 h_e / c_0$, where $c_0 = \sqrt{E/\rho}$ is the wave speed in a bar. Thus, the Courant dimensionless number C used for time integration was set as $C = \frac{c_0 \Delta t}{h_e} = 0.5$. It should be mentioned that the CFL (Courant-Friedrichs-Levy) condition for the linear finite element method with lumped mass matrices in the one-dimensional case reaches the value $C_{cr} = 1$.

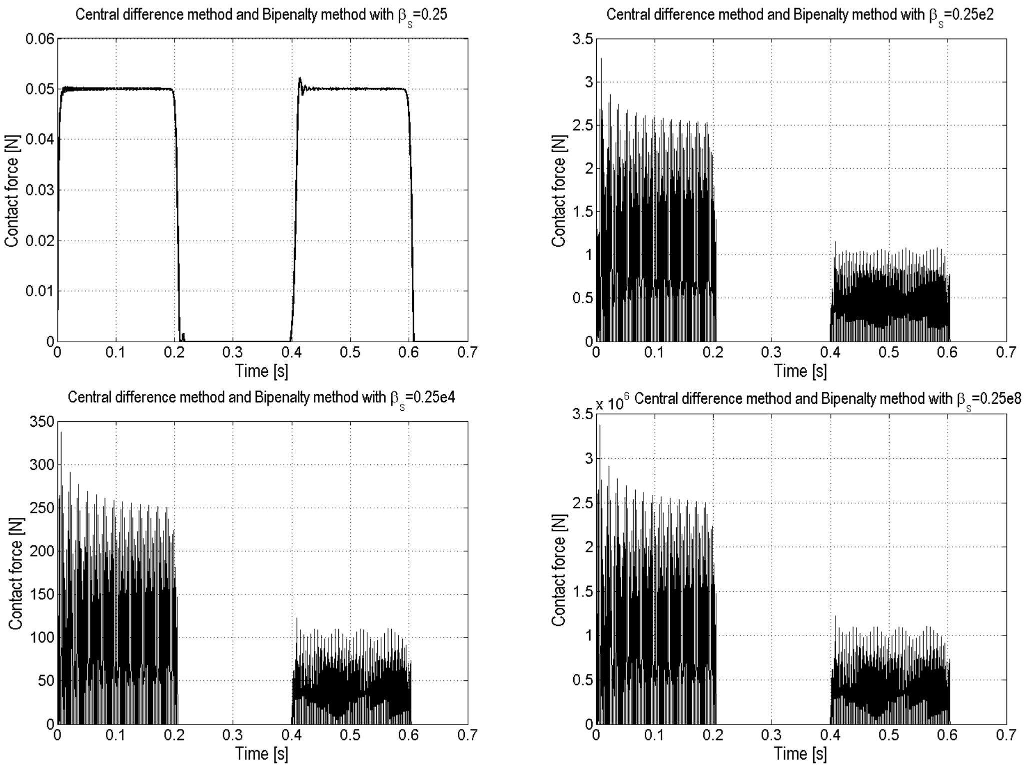

Figure 2. Time history of contact force for impact of two bars with different lengths - the central difference method with Courant number $C = 0.5$ for $\beta_s = \{0.25;\ 0.25e2;\ 0.25e4;\ 0.25e8\}$ and optimal bipenalty stabilization setting.

In Figures 2 and 3, one can see the time histories of contact forces between two elastic bars from Figure 1 computed by the central difference method and stabilized explicit scheme. In both cases, the bipenalty method with the optimal ratio of stiffness and mass penalty parameters were used. The time histories are presented for several values of dimensionless stiffness penalty parameter as follows $\beta_s = \{0.25;\ 0.25e2;\ 0.25e4;\ 0.25e8\}$. One can see that results for $\beta_s = 0.25$ for both used time schemes exhibit excellent progress, because this value of β_s corresponds to stiffness of the finite element in contact. On the other hand, the results of the central difference method for higher value of β_s shown significant spurious oscillations of contact forces, where force amplitudes grow up with value of stiffness penalty parameter β_s. In Figure 3, the results for the stabilized explicit scheme are presented for higher values of β_s. In principle, for higher β_s, one can see the contact force histories independent of β_s. Further, the stabilized explicit scheme produces robust and stable results for contact forces for a large range of stiffness penalty parameters including extremely higher values.

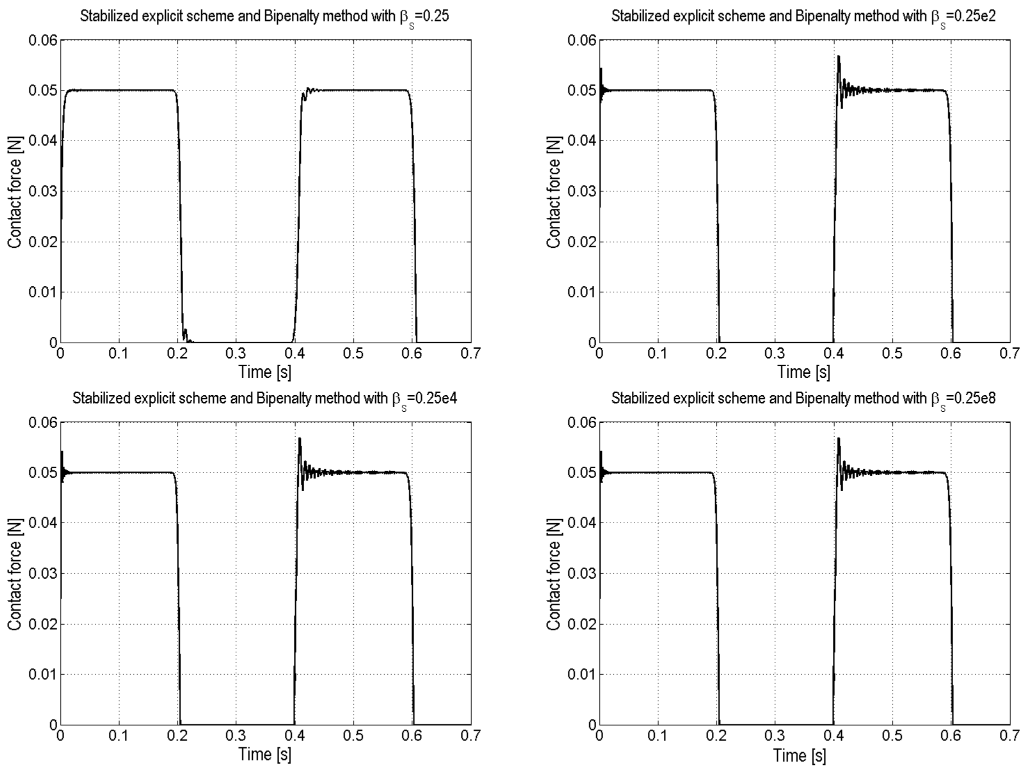

Figure 3. Time history of contact force for impact of two bars with different lengths - the stabilized explicit method with Courant number $C = 0.5$ for $\beta_s = \{0.25;\ 0.25e2;\ 0.25e4;\ 0.25e8\}$ and optimal bipenalty stabilization setting.

Conclusions

We have presented a numerical approach for finite element solution for contact-impact problems with stabilization of spurious contact oscillations. The stabilization is based on the bipenalty enforcement of contact constraints and modification of the explicit method with separation of bulk and contact accelerations. Based on the numerical test, we can conclude that the motivated approach is an efficient tool for accurate modeling of contact-impact problems with small pollution of contact spurious oscillations. The results obtained by the stabilized explicit scheme in connection with the bipenalty method are *less sensitive* to the choice of the penalty parameter in contrast to standard approach. In future work, we will focus on multidimensional implementation and testing and also on lumping of penalized additional mass matrix.

Acknowledgements

The work of R. Kolman was supported by the grant projects with No. 17-22615S of the Czech Science Foundation (CSF) and the work of J. Kopačka was supported by the grant projects with No. 16-03823S of CSF within institutional support RVO:61388998. The work of D. Gabriel was supported by the Centre of Excellence for Nonlinear Dynamic Behaviour of Advanced Materials in Engineering CZ.02.1.01/0.0/0.0/15_003/0000493 (Excellent Research Teams) in the framework of Operational Programme Research, Development and Education. The work was also supported by the the bilateral mobility project No. DAAD-16-12 and Projekt-ID 57219898 funded via DAAD from Federal Ministry of Education and Research.

References

[1] T. Belytschko, W. K. Liu, and B. Moran. *Nonlinear finite elements for continua and structures*. Wiley, Chichester, 2008.

[2] T. Belytschko and M. O. Neal. Contact-impact by the pinball algorithm with penalty and lagrangian methods. *International Journal for Numerical Methods in Engineering*, 31(3):547–572, 1991.

[3] N. J. Carpenter, R. L. Taylor, and M. G. Katona. Lagrange constraints for transient finite element surface contact. *International Journal for Numerical Methods in Engineering*, 32(1):103–128, 1991.

[4] P. Deuflhard, R. Krause, and S. Ertel. A contact-stabilized newmark method for dynamical contact problems. *International Journal for Numerical Methods in Engineering*, 73(9):1274–1290, 2008.

[5] J. Hetherington, A. Rodríguez-Ferran, and H. Askes. The bipenalty method for arbitrary multipoint constraints. *International Journal for Numerical Methods in Engineering*, 93(5):465–482, 2013.

[6] I. Huněk. On a penalty formulation for contact-impact problems. *Computers & Structures*, 48(2):193 – 203, 1993.

[7] C. Kane, E. Repetto, M. Ortiz, and J. E. Marsden. Finite element analysis of nonsmooth contact. *Computer Methods in Applied Mechanics and Engineering*, 180(1–2):1–26, Nov. 1999.

[8] H. B. Khenous, P. Laborde, and Y. Renard. Mass redistribution method for finite element contact problems in elastodynamics. *European Journal of Mechanics - A/Solids*, 27(5):918–932, Sept. 2008.

[9] J. Kopačka, A. Tkachuk, G. Gabriel, R. Kolman, M. Bischoff, and J. Plešek. On stability and reflection-transmission analysis of the bipenalty method in impact-contact problems: a one-dimensional, homogeneous case study. *International Journal for Numerical Methods in Engineering*, accepted, 2017.

[10] T. A. Laursen and V. Chawla. Design of energy conserving algorithms for frictionless dynamic contact problems. *International Journal for Numerical Methods in Engineering*, 40(5):863–886, 1997.

[11] Y. Renard. The singular dynamic method for constrained second order hyperbolic equations: Application to dynamic contact problems. *Journal of Computational and Applied Mathematics*, 234(3):906–923, June 2010.

[12] A. Tkachuk. *Variational methods for consistent singular and scaled mass matrices*. PhD thesis, Institut für Baustatik und Baudynamik der Universität Stuttgart, Stuttgart, 2013.

[13] P. Wriggers. *Computational contact mechanics*. Springer, Berlin ;;New York, 2nd ed. edition, 2006.

[14] S. R. Wu. A priori error estimates for explicit finite element for linear elasto-dynamics by galerkin method and central difference method. *Computer Methods in Applied Mechanics and Engineering*, 192(51):5329 – 5353, 2003.