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A scaled boundary NURBS approach for nonlinear solid analysis

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Micro Abstract
In this contribution, the scaled boundary formulation is proposed as a discretization technique which is based on the isogeometric concept. By this means, the representation of a solid body is given by the boundary surface of the body and a radial scaling parameter which is used to describe the interior. NURBS shape functions are employed to define the geometry as well as to approximate the solution field. The numerical examples are given for elasto-plastic material behavior at small strains.

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1 Introduction

In computer aided design (CAD), geometrical 3D models can be represented through a set of bounding surfaces as shown in Figure 1. By this means, a scaling center C is chosen in the interior domain which allows a decomposition of the 3D model in several subdomains known as sections. The same concept is also applicable to 2D geometries as shown in the examples of this contribution. A further property of CAD is the parametric description of surfaces and curves with NURBS functions. In the context of numerical mechanics, this property is exploited by isogeometric analysis (IGA) which was first introduced in [2]. In this analysis method, the solution field is interpolated by NURBS and CAD knot insertion serves as the discretization method. Based on the previously mentioned domain decomposition and the application of IGA for the numerical simulation, the scaled boundary approach [4] is introduced and applied in this contribution, see [3] and [1]. For the sake of simplicity, the method is studied for the case of 2D problems. For the same reason, the linear elastic context is given in the first formulations which is afterward extended to plasticity at small strains.

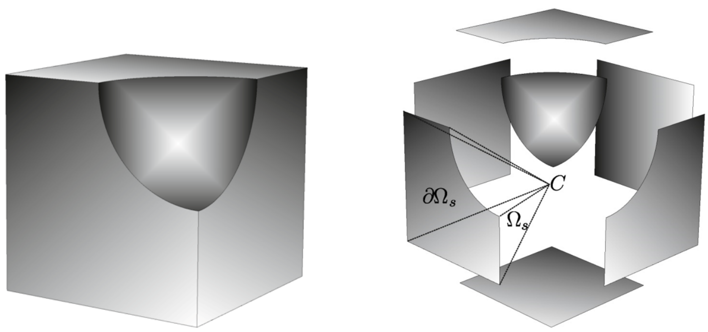

Figure 1. CAD modeling of a solid with boundary representation.

2 Parametrization

A 2D geometric CAD model can be decomposed in subdomains Ω_s by prescribing a scaling center C as shown in Figure 2. In a carthesian coordinate system ($\hat{x}$-$\hat{y}$), the position of C is given by $\hat{\boldsymbol{x}}_0$.

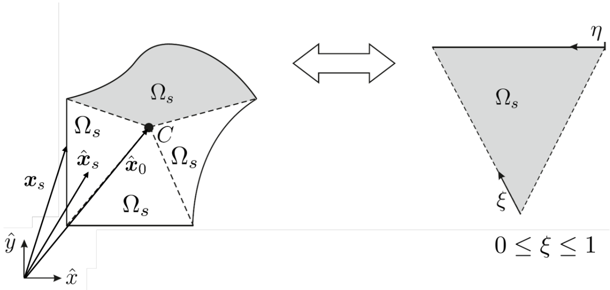

Figure 2. Parametrization of geometry.

Similarly to the geometric interpolation scheme used in FEM analysis, the boundary curve $\partial\Omega_s$ of a subdomain is given as a linear combination of shape functions and nodal coordinates,

$$\boldsymbol{x}_s = \boldsymbol{N}_b(\eta)\boldsymbol{X}_s. \tag{1}$$

Here $\boldsymbol{N}_b(\eta)$ is a matrix containing NURBS functions while $\boldsymbol{X}_s$ is a vector which contains the coordinates of the CAD control points. The parameter η is generally chosen to be $0 \leq \eta \leq 1$. For a point $\hat{\boldsymbol{x}}_s$ in the interior of the domain Ω_s a scaling parameter ξ with $0 \leq \xi \leq 1$ is introduced. Accordingly, the position of a point in Ω_s is given by

$$\hat{\boldsymbol{x}}_s = \hat{\boldsymbol{x}}_0 + \xi(\boldsymbol{x}_s(\eta) - \hat{\boldsymbol{x}}_0). \tag{2}$$

It should be noted that for $\xi = 0$ the point $\hat{\boldsymbol{x}}_s$ coincides with the scaling center $\hat{\boldsymbol{x}}_0$, while for $\xi = 1$ a boundary point $\boldsymbol{x}_s$ is addressed. All remaining kinematic relations, as e.g. the Jacobi matrix, which are necessary for the formulation of the boundary value problem (BVP) are basicly derived from (1) and (2).

3 Numerical approximation

NURBS basis functions are provided by the CAD model in order to represent boundary surfaces in 3D or boundary curves in 2D. For the boundary (circumferential) direction they are defined by

$$R_{i,p}(\eta) = \frac{N_{i,p}(\eta)w_i}{\sum_{k=1}^{n_{bc}} N_{k,p}w_k}. \tag{3}$$

The functions $N_{i,p}(\eta)$ are recursively defined B-Spline functions of order p while w_i are corresponding weight factors for the control points. The number of control points of the boundary of a specific subdomain (grey area in Figure 3) is denoted by n_{bc}. They are represented by an orange surrounding in Figure 3. The NURBS functions in scaling (radial) direction are defined in the same manner,

$$R_{j,q}(\xi) = \frac{N_{j,q}(\xi)w_j}{\sum_{m=1}^{n_{cp}} N_{m,q}w_m}, \tag{4}$$

in which n_{cp} is the number of control points in scaling direction (blue points in Figure 3). The boundary curve of a 2D geometry is obtained by the linear combination of NURBS and the coordinates $\hat{\boldsymbol{X}}_i$ of the control points,

$$\bar{\boldsymbol{x}}_s = \sum_{i=1}^{n_{bc}} R_{i,p}(\eta)\hat{\boldsymbol{X}}_i. \tag{5}$$

The same ansatz, which represents the backbone of IGA, is also applicable to the displacement field of the boundary, thus

$$\bar{\boldsymbol{u}}_s = \sum_{i=1}^{n_{bc}} R_{i,p}(\eta)\hat{\boldsymbol{U}}_i, \tag{6}$$

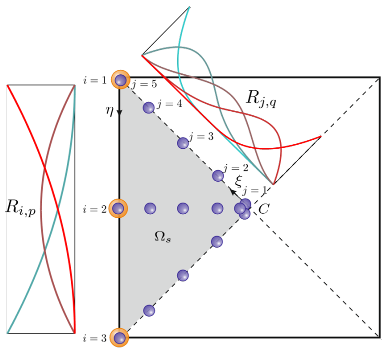

Figure 3. Description of geometry with NURBS.

in which $\hat{\boldsymbol{U}}_i$ are the displacements associated to the control points of the boundary, however they are functions of the scaling parameter ξ. Accordingly, the vectors $\hat{\boldsymbol{U}}_i$ are arranged in the vector $\boldsymbol{U}_s$ which is given as

$$\boldsymbol{U}_s(\xi) = \boldsymbol{N}_s(\xi)\boldsymbol{U}_j \tag{7}$$

The displacement response of a given subdomain is approximated by,

$$\boldsymbol{u}(\xi,\eta) = \boldsymbol{N}_b(\eta)\boldsymbol{U}_s(\xi). \tag{8}$$

Substituting (7) in (8) provides the displacement field in dependency of the basis functions in both directions,

$$\boldsymbol{u}(\xi,\eta) = \boldsymbol{N}_b(\eta)\boldsymbol{N}_s(\xi)\boldsymbol{U}_j, \tag{9}$$

in which $\boldsymbol{N}_b(\eta)$ and $\boldsymbol{N}_s(\xi)$ are matrices containing the NURBS shape functions.

4 Numerical examples

4.1 Linear analysis

The first example regards the benchmark of the infinite plate with circular hole under uniaxial loading. For the numerical simulation, the geometry of the plate is defined as shown in Figure 4. The stress fields from the analytical solution in are applied on the Neumann boundary while

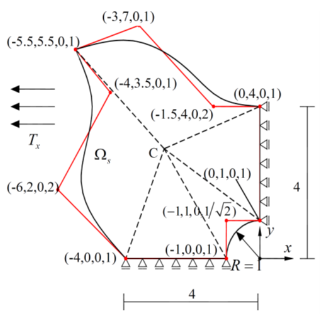

Figure 4. Plate with circular hole.

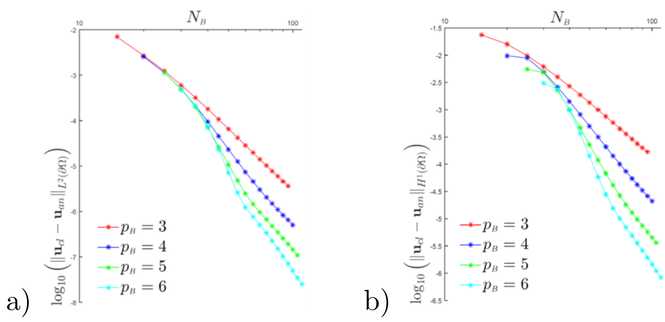

Figure 5. Convergence rates, a) L_2 Norm and b) H_1 Norm.

symmetry conditions are prescribed on the displacement boundary. The polynomial order of the NURBS functions are set identical in scaling and circumferential direction, $p = q$. The results of the L_2 and H_1 error norm from the comparison of the analytical and numerical displacement field are shown in Figure 5. The results shown the convergence behavior of the mesh by h-refinement as the number of control points increases. Furthermore, a p-refinement leads to higher convergence rates.

4.2 Nonlinear analysis

The nonlinear analysis considers a plate with circular hole for elasto-plastic material behavior at small strains, $\boldsymbol{\sigma} = \mathbb{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$. The boundary conditions are identical to the ones of the linear analysis in section 4.1, see Figure 6 (left). The material parameter are: Young's modulus $E = 100$, Poisson's ratio $\nu = 0.3$, thickness $h = 1$, yield stress $\sigma_y = 5$ and linear hardening $H = 10$. The polynomial orders are set for both directions to $p = q = 6$. The total number of control points for the boundary is defined as N_B and the control points per line in the scaling direction as N_C. The number of radial control points is set as $N_C \geq ceil(N_B/4) + p_C$ as suggested in [1]. Elasto-plastic analysis is performed for a fixed discretization with $N_B = 50$ and $N_C = 20$. For the comparison of the results, a standard FEM simulation is considered. A finite element mesh with linear isoparametric elements and the same degrees of freedom is employed for the comparison. The deformed meshes for the FEM analysis and the proposed scaled boundary NURBS approach are depicted in Figure 6 (middle). To compare the results of both simulations, the horizontal displacement at the upper left corner is evaluated. The displacement of the considered point in dependency of the applied load are shown in Figure 6 (right). The result of the proposed approach is in very good agreement with the solution of the finite element method. Further studies on the choice of p, N_B, q and N_C for optimal convergence with respect to FEM are subject of future research.

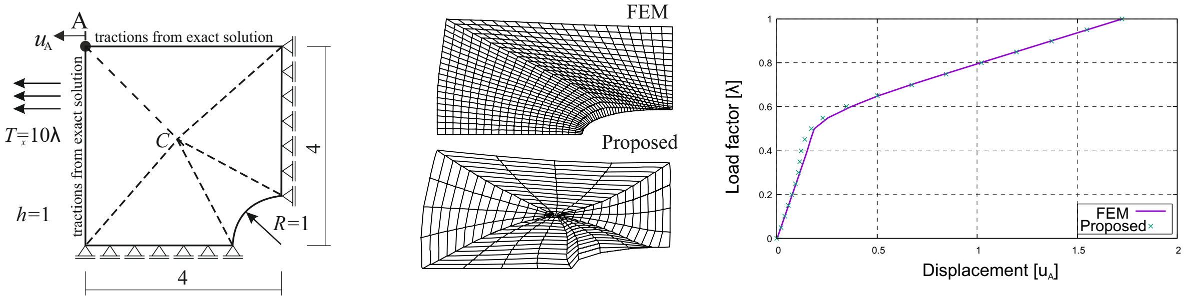

Figure 6. Left: Problem geometry and loading. Middle: Deformed plate. Right: Load displacement curve.

5 Conclusions

It has been shown that the proposed formulation is perfectly suitable for the boundary representation modeling technique in CAD. It combines the isogeometric analysis with the scaled-boundary finite-element formulation and it is extendable for nonlinear material behavior. Further developments include the extension to geometrical nonlinearity and 3D analysis.

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