Proceedings of the 7th GACM Colloquium on Computational Mechanics
for Young Scientists from Academia and Industry
October 11-13, 2017 in Stuttgart, Germany

Determination of optimal damping for passive control of vibration based on the design of limit cycles

Rafael A. Rojas[1]*, Erich Wehrle[1] and Renato Vidoni [1]

Micro Abstract
The optimal design of passive vibration control is a challenge for both application and research. These design methods are based on of structure optimization and models are typically solved in frequency domain. This work explores the benefits of introducing state-space methods on passive control. We propose an optimization approach based on the design of the limit cycles of mechanical systems under periodic forces. We exploit the analogy between damping optimization and energy harvesting to address simultaneously both technologies. An example of energy harvesting optimization is presented.

[1]Faculty of Science and Technology, Free University of Bozen-Bolzano, Bolzano, Italy
***Corresponding author**: rafaelangel.rojascordova@unibz.it

Introduction

Control of vibrations is common practice in both mechanical [2, 11] and civil engineering [10] to improving the performance of mechanical structures. When such control is done by adding passive elements as masses, springs or dampers, we speak of passive control of vibrations. Among all the structure optimization problems related to mechanical engineering, those of energy harvesting and damping design for passive control are of great relevance. In fact, the regulation of the dissipated energy is the key physical phenomenon that drives both technologies. In this paper we present a general formulation for optimization problems which allows to address both problems, damping optimization and energy harvesting within the same framework.

Given the complexity of formulating such optimization problems, several approaches have been proposed. We briefly classify those methods in (i) frequency domain approaches (e.g. [9]), (ii) time domain or state space approaches (e.g. [4, 5]) and (iii) inverse eigenvalue problems (e.g. [3]). In this paper we propose a state-space damping optimization method for linear mechanical systems. We achieve to combine the benefits of state-space methods and frequency-domain methods thanks to an asymptotic analysis. We formulate an optimization problem of a generic infinite-horizon integral objective function of the form

$$P = \int_0^\infty L(\underline{u}) \ \mathsf{d}t, \tag{1}$$

where $\underline{u}$ is the state variable of our mechanical system and L is the Lagrangian. To address this optimization problem, we leverage on the proof that under certain assumptions, a linear visco-elastic mechanical system under a periodic force with fundamental period T_0 converges asymptotically to a limit cycle with the same fundamental period. This allows addressing the objective function (1) in the steady-state of the system even in cases when that integral diverges. We derive an explicit analytical expression $\hat{\underline{u}}(t, x)$ of the limit cycle, where $\underline{x}$ is the vector of design variables and $t \in [0, T_0]$ is a time-like parameter.

Model of the vibrating system

Consider a generic linear mechanical system with n degrees of freedom, governed by the equation

$$\underline{\underline{m}}\,\ddot{\underline{q}} + \underline{\underline{d}}(\underline{x})\,\dot{\underline{q}} + \underline{\underline{k}}\,\underline{q} = \underline{F}(t), \tag{2}$$

where $\underline{\underline{m}}$ and $\underline{\underline{k}}$ are the mass and stiffness $n \times n$ matrices, respectively, $\underline{F}(t) \in \mathbb{R}^n$ is the generalized external forces vector, and $\underline{q} \in \mathbb{R}^n$ is the vector of generalized coordinates of the system. The matrix $\underline{\underline{d}}(x)$ is damping matrix, which here depends on $\underline{x}$, the design variables. To write the system (2) in the state-space we introduce the state vector $\underline{u} \in \mathbb{R}^{2n}$ defined as $\underline{u} = \begin{bmatrix}\underline{q}^T & \dot{\underline{q}}^T\end{bmatrix}^T$. Therefore, the system (2) has the following form in the state-space

$$\dot{\underline{u}} = \underline{\underline{A}}(\underline{x})\underline{u} + \underline{f}(t), \tag{3}$$

where

$$\underline{\underline{A}}(\underline{x}) = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{I}}_n \\ -\underline{\underline{m}}^{-1}\underline{\underline{k}} & -\underline{\underline{m}}^{-1}\underline{\underline{d}}(x) \end{bmatrix}, \tag{4}$$

where $\underline{\underline{I}}_n$ is the $n \times n$ identity matrix and $\underline{f} = \begin{bmatrix}\underline{0} & \underline{F}^T\end{bmatrix}^T$. System (3) has the well known solution

$$\underline{u}(t, \underline{x}) = \underline{\underline{\Phi}}(t, \underline{x})\underline{u}_i + \int_0^t \underline{\underline{\Phi}}(t - s, \underline{x})\underline{f}(s)\mathsf{d}s, \tag{5}$$

where $\underline{u}_i$ are the initial condition of the system, and $\underline{\underline{\Phi}}$ is the state transition matrix associated to the homogeneous solution of the linear equation (3), defined by

$$\underline{\underline{\Phi}}(t, \underline{x}) = e^{\underline{\underline{A}}(\underline{x})t}. \tag{6}$$

This solution lacks of the simplicity of traditional analysis, where the steady state solution is derived from the linear superposition of the normal modes, but it has the outstanding benefit of being a general solution of (2) without any kind of suppositions on such a mechanical system. In particular we exploit two good properties of this solution. Firstly, we are not constrained to use a particular form of damping, e.g., proportional damping. Secondly, we can derive general conditions for the existence of limit cycles from the stability of the system. However, at a first glance, the expression (5) makes difficult a direct analysis of the steady-state behaviour of the system. In the next section we prove that this is possible without the use of Fourier of Laplace transform, remaining in the time domain.

Determination of the limit cycle

A necessary property of the mechanical system (2) to converge to a limit cycle is the asymptotic stability of the homogeneous system associated to (3). A sufficient condition for that is the three matrices $\underline{\underline{m}}$, $\underline{\underline{d}}(\underline{x})$ and $\underline{\underline{k}}$ are positive definite [6]. If, further, the matrix $\underline{\underline{A}}(\underline{x})$ is diagonalizable, the state transition matrix has the following properties:

$$\lim_{t \to \infty} \underline{\underline{\Phi}}(t, \underline{x}) = \underline{\underline{0}} \tag{7}$$

$$\|\underline{\underline{\Phi}}(t, \underline{x})\| < 1 \qquad \forall t > 0. \tag{8}$$

These two properties of the state transition matrix allow us to state the following theorem which prove may be found in [8]:

Theorem 1 *Consider the system (3) under periodic force $\underline{f}(t)$ with fundamental period T_0. If the state transition matrix of such a system has the properties* (7) *and* (8), *for every initial condition, the system converges to the limit cycle of period T_0 described by:*

$$\hat{\underline{u}}(t, \underline{x}) = \underline{\underline{\Phi}}(t, \underline{x})\left[\hat{\underline{u}}_0(\underline{x}) + \int_0^t \underline{\underline{\Phi}}(-s, \underline{x})\underline{f}(s)\mathsf{d}s\right] \qquad \forall t \in [0,\, T_0], \tag{9}$$

where

$$\underline{\hat{u}}_0(\underline{x}) = \int_0^{T_0} \underline{\underline{R}}(\underline{x}) \left(\underline{\underline{I}}_{2n} - \underline{\underline{R}}(\underline{x})\right)^{-1} \underline{\underline{\Phi}}(-s, \underline{x}) \underline{f}(s) \mathsf{d}s \tag{10}$$

$\underline{\underline{I}}_{2n}$ *is the* $2n \times 2n$ *identity matrix and* $\underline{\underline{R}}(\underline{x}) = \underline{\underline{\Phi}}(T_0, \underline{x})$.

This theorem permits the approximation of the objective function (1) in cases when the contribution made by the steady state of the system is so large that the contribution of the transitory is negligible. In these cases, the contribution of the steady state to (1) is grasped by the contribution of the limit cycle (9). To formulate the numerical optimization problem it is sufficient to introduce a time partition $t_1, t_2, ..., t_N$ in the limit cycle. Then, on the one hand, the objective function is approximated by a Gauss quadrature,

$$\hat{P}(x) \approx \sum_{i=1}^{N} w_i L\left(\hat{u}(t_i, x)\right). \tag{11}$$

On the other hand, any constraint on the state variable $g_i(\underline{u}) \leq 0$ is mapped to N constraints on each point of the time partition,

$$g_i(\underline{\hat{u}}) \leq 0 \implies g_i(\underline{\hat{u}}(t_1, x)) \leq 0,\ g_i(\underline{\hat{u}}(t_2, x)) \leq 0, ...,\ g_i(\underline{\hat{u}}(t_N, x)) \leq 0. \tag{12}$$

We end this section deriving an analytical expression of (9). Such equations are defined in the interval $[0, T_0]$, so we can write the periodic force $\underline{f}(t)$ as a Fourier series of an odd function

$$\underline{f}(t) = \sum_{k=0}^{\infty} \underline{\hat{f}}_k \sin(k\omega_0 t). \tag{13}$$

We have to evaluate the integral in the equation (9), which has the following form:

$$\sum_{k=1}^{\infty} \left(k^2\omega_0^2 \underline{\underline{I}}_{2n} + \underline{\underline{A}}^2(\underline{x})\right)^{-1} \left[k\omega_0 \underline{\underline{I}}_{2n} - \underline{\underline{\Phi}}(-t, \underline{x}) \left(k\omega_0 \cos(k\omega_0 t) \underline{\underline{I}}_{2n} + \sin(k\omega_0 t) \underline{\underline{A}}(\underline{x})\right)\right] \underline{\hat{f}}_k. \tag{14}$$

Substituting (14) in (9) and (10), we get

$$\underline{\hat{u}}\left(t, \underline{x}\right) = -\sum_{k=1}^{\infty} \left(k^2\omega_0^2 \underline{\underline{I}}_{2n} + \underline{\underline{A}}^2(\underline{x})\right)^{-1} \left(k\omega_0 \underline{\underline{I}}_{2n} \cos\left(k\omega_0 t\right) + \underline{\underline{A}}(\underline{x}) \sin\left(k\omega_0 t\right)\right) \underline{\hat{f}}_k. \tag{15}$$

An example of energy harvesting optimization

In this section, we address the problem of choosing the optimal combination of dampers which maximize the harvested energy from the Euler-Bernoulli beam placed in a bed of dampers (c.f. Figure 1). The beam in Figure 1 has constant section and density, and is pinned at both ends. The position of each point of the neutral axis of the beam from the left is given by z. The beam is under the action of a punctual periodic force $p(t)$ at z_f and each damper with damping coefficient c_i is placed at z_i. The equation of motion of such a system is:

$$EI\frac{\partial^4 w}{\partial z^4} + c_0\frac{\partial w}{\partial t} + \rho A\frac{\partial^2 w}{\partial t^2} = p(t)\delta(z - z_f) - \sum_{i=0}^{m} c_i w(z,t)\delta(z - z_i), \tag{16}$$

where E, I, ρ, A, w and c_0 are respectively the Young modulus, the second moment of area about the neutral axis, the density, the cross-section of area, the vertical displacement of the neutral axis and the internal damping of the beam's material. The symbol $\delta(z)$ represents a Dirac's delta. We further introduce the following variables:

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}, \qquad \zeta_0 = \frac{c_0\omega_1}{2\rho A}, \qquad \zeta_i = \frac{c_i\omega_1}{\rho A}, \qquad \tau = \omega_1 t,$$

$$\xi = \frac{z}{L}, \qquad v(\tau) = \frac{2p(\tau)L^3}{EI}, \qquad \eta(\tau,\xi) = \frac{w(\tau,\xi)}{L}. \tag{17}$$

We solve (16) using the change of variables (17) with the Galerkin method approximating the non-dimensional displacement as

$$\eta(\tau,\xi) = \sum_{k=1}^{n} \varphi_k(\xi) q_k(\tau), \tag{18}$$

where $\varphi_k(\xi)$ $k = 1, \ldots, n$ are a set of orthonormal functions that satisfy the boundary conditions of the problem. Then, we obtain the discretized equations of motion in the generalized coordinates $\underline{q} \in \mathbb{R}^n$:

$$\ddot{\underline{q}} + \Big(\underline{\underline{d}}_0 + \sum_{i=1}^{m} \underline{\underline{d}}_i\Big)\dot{\underline{q}} + \underline{\underline{k}}\ \underline{q} = \underline{F}(\tau), \tag{19}$$

where each component of $\underline{F}$ and $\underline{\underline{d}}_i$ are respectively $\underline{F}_k = \varphi_i(\xi_f)v(\tau)$ and $(\underline{\underline{d}}_i)_{kj} = 2\zeta_i\varphi_k(\xi_i)\varphi_j(\xi_i)$, and $\underline{\underline{d}}_0$ and $\underline{\underline{k}}$ are two diagonal matrices with components $(\underline{\underline{d}}_0)_{kk} = 2\zeta_0$ and $(\underline{\underline{k}})_{kk} = k^4$.

For m dampers we define our optimization variable $x \in \mathbb{R}^{2m}$ as $x = [\zeta_1, \cdots, \zeta_m, \xi_1, \cdots, \xi_m]$. The total power drained by the bed of dampers is given by

$$L(\underline{x}) = \dot{\underline{q}}^T \left(\sum_{i=1}^{m} \underline{\underline{d}}_i(x_i) \right) \dot{\underline{q}}. \tag{20}$$

Given the definition our design variable $\underline{x}$, we have to introduce two bounds on it to preserve its physical meaning. First, the damping coefficient of each damper must be positive, we set $0 \leq x_i \leq \zeta_{\max i}$ $i = 1, \cdots, m$. Secondly, we need to constraint the position of the dampers on the beam, so we set $(i-1)/m \leq x_{i+m} \leq i/m,\ i = 1, \cdots, m$. Finally, we introduce the constraint of limiting the displacement of the beam, i.e., $|\eta(\tau,\xi)| \leq \eta_0$ on the limit cycle. The

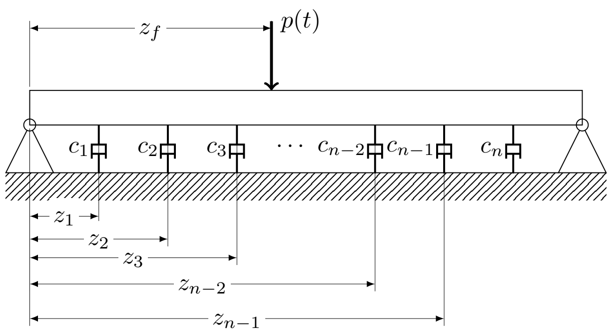

Figure 1. Beam on a damper bed.

optimization was carried out using an upper bound to the dimensionless displacement $\eta_0 = 0.1$ and a dimensionless force $v(\tau) = 0.4\sin(\tau)$. We used the second-order algorithm NLPQLP [1,7], converging in 14 iterations and 109 evaluations from the starting point shown in the following table. Further initial designs were investigated that all resulted in the same optimal design, needing between 12 and 22 iterations. The results and bounds on the design variable are reported in Table 1, where we can appreciate the symmetry of the solution. In fact, that solution reflexes the influence of the first mode of the system and corresponds to the expected behaviour given the dimensionless force that we have used.

Conclusion and Outlook

This paper has presented a damping optimization method based in the design of the limit cycle of a linear mechanical system under a periodic force. We gave sufficient conditions for the existence of such limit cycle and provide a method to formulate the objective function and the constraints of the optimization problem. As example, a problem of energy harvesting was developed. In the

Table 1. Optimization results of energy-harvesting beam (all measures are dimensionless)

Parameter	Initial design $\underline{x}^0$	Lower bound $\underline{x}^L$	Upper bound x^U	Optimum x^*
Damping coefficient ζ_1	0.5005	0.001	1.000	0.664
Damping coefficient ζ_2	0.5005	0.001	1.000	1.000
Damping coefficient ζ_3	0.5005	0.001	1.000	0.670
Position ξ_1	0.167	0.001	0.333	0.333
Position ξ_2	0.5	0.333	0.666	0.500
Position ξ_3	0.832	0.666	0.999	0.667
Energy harvested P	–	–	–	0.126
Maximum displacement η_1	–	–	–	0.071
Maximum displacement η_2	–	–	–	0.100
Maximum displacement η_3	–	–	–	0.071

future, this problem may be improved to handle more general kinds of problems. For example linear systems under quasi-periodic and stochastic excitation may be further step which may be addressed from the actual framework. In such cases the existence of limit cycles is not assured, but we can extent the current analysis to global attractors.

References

[1] Y.-H. Dai and K. Schittkowski, *A sequential quadratic programming algorithm with non-monotone line search*, Pacific Journal of Optimization, 4 (2008), pp. 335–351.

[2] D. DeBra, *Vibration isolation of precision machine tools and instruments*, CIRP Annals-Manufacturing Technology, 41 (1992), pp. 711–718.

[3] P. Lancaster and U. Prells, *Inverse problems for damped vibrating systems*, Journal of Sound and Vibration, 283 (2005), pp. 891–914.

[4] O. Lavan and R. Levy, *Optimal peripheral drift control of 3d irregular framed structures using supplemental viscous dampers*, Journal of Earthquake Engineering, 10 (2006), pp. 903–923.

[5] J. Lee, A. H. Ghasemi, C. E. Okwudire, and J. Scruggs, *A linear feedback control framework for optimally locating passive vibration isolators with known stiffness and damping parameters*, Journal of Vibration and Acoustics, 139 (2017).

[6] D. R. Merkin, *Introduction to the Theory of Stability*, vol. 24, Springer Science & Business Media, 2012.

[7] K. Schittkowski, *NLPQLP: A fortran implementation of a sequential quadratic programming algorithm with distributed and non-monotone line search*, user's guide, version 3.1, Department of Computer Science, University of Bayreuth, 2010.

[8] J. Slane and S. Tragesser, *Analysis of periodic nonautonomous inhomogeneous systems*, Nonlinear Dynamics and Systems Theory, 11 (2011), pp. 183–198.

[9] I. Takewaki, *Optimal damper placement for minimum transfer functions*, Earthquake Engineering & Structural Dynamics, 26 (1997), pp. 1113–1124.

[10] J. K. Whittle, M. S. Williams, T. L. Karavasilis, and A. Blakeborough, *Optimal placement of viscous dampers for seismic building design*, in Design Optimization of Active and Passive Structural Control Systems, N. D. Lagaros, N. D. Lagaros, and C. C. Mitropoulou, eds., Information Science Reference, 2012.

[11] Y. Yu, N. G. Naganathan, and R. V. Dukkipati, *A literature review of automotive vehicle engine mounting systems*, Mechanism and machine theory, 36 (2001), pp. 123–142.