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A phase field model for porous plastic solids at ductile fracture

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Micro Abstract
This work outlines a variational framework for the phase field modeling of fracture in porous plastic solids. The phase field regularizes sharp crack surfaces by a specific gradient damage formulation. A model for porous plasticity with a growth law for the evolution of the void fraction is developed and linked to a failure criterion in terms of the elastic-plastic work density. It is shown that this approach is able to model basic phenomena of ductile failure such as cup-cone failure surfaces.

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Constitute framework for porous plastic solids at fracture

The presented model is formulated in context of finite deformations. It incorporates a gradient plasticity theory and the crack propagation is modeled with the phase field approach. Hence the global fields can be identified as the deformation map $\boldsymbol{\varphi}$, the phase field variable d and the global hardening variable α

$$\boldsymbol{\varphi} : \begin{cases} \mathcal{B} \times \mathcal{T} \to \mathcal{R}^3 \\ (\boldsymbol{X}, t) \mapsto \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) \end{cases} \qquad d : \begin{cases} \mathcal{B} \times \mathcal{T} \to [0,1] \\ (\boldsymbol{X}, t) \mapsto d(\boldsymbol{X}, t) \end{cases} \qquad \alpha : \begin{cases} \mathcal{B} \times \mathcal{T} \to \mathcal{R} \\ (\boldsymbol{X}, t) \mapsto \alpha(\boldsymbol{X}, t) \end{cases} . \tag{1}$$

Here the phase field variable denotes with $d = 0$ the unbroken solid and $d = 1$ the fractured state. The decomposition of the deformation into an elastic and plastic part is based on the multiplicative split of the deformation gradient $\boldsymbol{F} := \nabla\boldsymbol{\varphi} = \boldsymbol{F}^e\boldsymbol{F}^p$, as introduced in [2], and the introduction of a *Lagrangian plastic metric* $\boldsymbol{G}^p \in Sym_+(3)$ and an elastic deformation measure in form of the *Eulerian finger tensor* $\boldsymbol{b}^e = \boldsymbol{F}^e\boldsymbol{F}^{eT}$. This gives the definitions

$$\boldsymbol{b}^e := \boldsymbol{F}\boldsymbol{G}^{p-1}\boldsymbol{F}^T \quad \text{with} \quad \boldsymbol{G}^p(\boldsymbol{X}, t_o) = \boldsymbol{1} \quad \text{and} \quad \boldsymbol{h}^e = \tfrac{1}{2}\ln[\boldsymbol{b}^e], \tag{2}$$

where $\boldsymbol{h}^e$ is the *Eulerian elastic Hencky tensor*. The evolution of the plastic metric is given by the *Eulerian plastic rate of deformation tensor* introduced in [5],

$$\boldsymbol{d}^p := -\tfrac{1}{2}\,(\pounds_{\boldsymbol{v}}\boldsymbol{b}^e)\boldsymbol{b}^{e-1} = \boldsymbol{F}\boldsymbol{D}^p\boldsymbol{F}^{-1} \quad \text{with} \quad \boldsymbol{D}^p = -\tfrac{1}{2}\,\dot{\boldsymbol{G}}^{p-1}\boldsymbol{G}^p. \tag{3}$$

To account for the microstructural changes in terms of the porosity in the material, the *void volume fraction* f is introduced. It and its evolution are expressed in terms of the determinant of the deformation gradient $J := \det[\boldsymbol{F}]$, see also [1]. The initial void volume fraction is denoted as f_0.

$$\dot{f} = (1-f)\frac{\dot{J}}{J} \quad \Rightarrow \quad f = \hat{f}(J) = \max\left[f_0, 1 - \frac{1-f_0}{J}\right]. \tag{4}$$

Pseudo Energetic Response

The pseudo energetic response is characterized by the constitutive state $\mathfrak{C}$ given by

$$\mathfrak{C} := \{\nabla\boldsymbol{\varphi}, \boldsymbol{G}^{p-1}, \alpha, \nabla\alpha, d, \nabla d\}. \tag{5}$$

Based on that, one can formulate the elastic-plastic fracture work density

$$W(\mathfrak{C}) = g(d)[w^e(\boldsymbol{h}^e) + w^p(\alpha, \nabla\alpha)] + [1 - g(d)]w_c + w^f(d, \nabla d), \tag{6}$$

in terms of an elastic $w^e(\boldsymbol{h}^e)$, plastic $w^p(\alpha, \nabla\alpha)$ and a phase field part $w^f(d, \nabla d)$. Note that both the elastic as well as the plastic contribution within the work density function are degraded by the degradation function $g(d) = (1 - d)^2$. The material parameter w_c denotes the *critical work density* of the solid controls the onset of damage. The elastic work density is assumed to have the simple quadratic form

$$w^e(\boldsymbol{h}^e) = \frac{\kappa}{2}\,\mathrm{tr}^{\,2}[\boldsymbol{h}^e] + \mu\,\mathrm{tr}[(\mathrm{dev}[\boldsymbol{h}^e])^2]. \tag{7}$$

The plastic work density is expressed in terms of a saturation-type hardening function $\hat{y}_M(\alpha)$, the plastic length scale $l_p > 0$, the initial yield stress y_0 and the gradient of α

$$w^p(\alpha, \nabla\alpha) = \int_0^\alpha \hat{y}_M(\widetilde{\alpha})\, d\widetilde{\alpha} + y_0 \frac{l_p^2}{2}||\nabla\alpha||^2. \tag{8}$$

Following [4] the fracture part of the work density is based on a geometric regularization of sharp crack continuities and takes the form

$$w^f(d, \nabla d) = 2\frac{w_c}{\zeta} l_f \gamma_l(d, \nabla d) \quad \text{with} \quad \gamma_l(d, \nabla d) = \frac{1}{2l_f} d^2 + \frac{l_f}{2}||\nabla d||^2, \tag{9}$$

in terms of the *crack surface density function* γ_l, the fracture slope parameter ζ and the fracture length scale l_f.

Dissipative Response

The dissipative response on the plastic side is characterized by the *plastic yield function* ϕ^p which is expressed in the Kirchhoff stress space and derived from the classical Gurson yield hypersurface for a porous material

$$\phi^p(f^p, r^p; f) = \sqrt{\|\,\mathrm{dev}[\boldsymbol{f}^p]\|^2 + \tfrac{1}{6} f (\mathrm{tr}[\boldsymbol{f}^p])^2} - \sqrt{\tfrac{2}{3}}\, r^p, \tag{10}$$

where we introduced the plastic driving force $\boldsymbol{f}^p := \boldsymbol{\tau}$ and the plastic resistance force $r^p := \partial_\alpha W - \mathrm{Div}(\partial_{\nabla\alpha} W)$. The initiation of fracture is determined by a *fracture threshold function* in terms of the fracture driving force f^f and the fracture resistance r^f

$$\phi^f(f^f - r^f) := f^f - r^f := \partial_d W - \mathrm{Div}[\partial_{\nabla d} W]. \tag{11}$$

With these dissipative functions defined one can introduce a *dual dissipation potential function* D^* describing a viscous regularized evolution in terms of the dual constitutive state $\mathfrak{F}$

$$D^*(\mathfrak{F}) = \frac{3}{4\eta_p}\,\langle \phi^p(\boldsymbol{f}^p, r^p; f)\rangle^2 + \frac{1}{2\eta^f}\langle \phi^f(f^f - r^f)\rangle^2 \quad \text{with} \quad \mathfrak{F} := \{\boldsymbol{f}^p, r^p, f^f - r^f\}. \tag{12}$$

Extended Variational Principle

Based on the introduced work density and the dual dissipation potential function, the global extended rate potential for gradient plasticity coupled with gradient damage mechanics reads

$$\Pi^*(\dot{\boldsymbol{\varphi}}, \dot{\alpha}, \dot{d}, \boldsymbol{d}^p, \mathfrak{F}) = \int_{\mathcal{B}} \pi^*(\dot{\mathfrak{C}}, \mathfrak{F}; \mathfrak{C})\, dV \;-\; \Pi_{ext}(\dot{\boldsymbol{\varphi}}), \tag{13}$$

where π^* is the mixed potential density defined as

$$\pi^*(\dot{\mathfrak{C}}, \mathfrak{F}; \mathfrak{C}) = \frac{d}{dt} W(\mathfrak{C}) + \boldsymbol{f}^p : \boldsymbol{d}^p - r^p \cdot \dot{\alpha} + (f^f - r^f)\cdot \dot{d} - D^*(\mathfrak{F}). \tag{14}$$

The evolution problem is fully governed by the variational principle

$$\{\dot{\boldsymbol{\varphi}}, \dot{\alpha}, \dot{d}, \boldsymbol{d}^p, \boldsymbol{\mathfrak{F}}\} = \mathrm{Arg}\{ \inf_{\dot{\boldsymbol{\varphi}},\dot{\alpha},\dot{d},\boldsymbol{d}^p} \; \sup_{\boldsymbol{\mathfrak{F}}} \; \Pi^*(\dot{\boldsymbol{\varphi}}, \dot{\alpha}, \dot{d}, \boldsymbol{d}^p, \boldsymbol{\mathfrak{F}})\}. \tag{15}$$

The corresponding Euler equations of this variational principle read

$$\begin{array}{lllcl}
1. & \textit{Stress equilibrium} & \delta_{\dot{\boldsymbol{\varphi}}}\pi^* & \equiv & -\mathrm{Div}\,[\,\partial_{\nabla\boldsymbol{\varphi}} W\,] = \bar{\boldsymbol{\gamma}}_0 \\
2. & \textit{Hardening force} & \delta_{\dot{\alpha}}\pi^* & \equiv & \partial_{\alpha} W - \mathrm{Div}\,[\,\partial_{\nabla\alpha} W\,] - r^{\alpha} = 0 \\
3. & \textit{Fracture force} & \delta_{\dot{d}}\pi^* & \equiv & \partial_{d} W - \mathrm{Div}\,[\,\partial_{\nabla d} W\,] + f^{f} - r^{f} = 0 \\
4. & \textit{Plastic force} & \partial_{\boldsymbol{d}^p}\pi^* & \equiv & \partial_{\boldsymbol{h}^e} W + \boldsymbol{f}^p = \boldsymbol{0} \\
5. & \textit{Plastic deformation} & \partial_{\boldsymbol{f}^p}\pi^* & \equiv & \boldsymbol{d}^p - \partial_{\boldsymbol{f}^p} D^* = \boldsymbol{0} \\
6. & \textit{Equivalent strain} & \partial_{r^p}\pi^* & \equiv & -\dot{\alpha} - \partial_{r^p} D^* = 0 \\
7. & \textit{Fracture phase field} & \partial_{f^f - r^f}\pi^* & \equiv & \dot{d} - \partial_{f^f - r^f} D^* = 0
\end{array} \tag{16}$$

along with the Neumann-type boundary conditions of the form

$$(\partial_{\nabla\boldsymbol{\varphi}} W)\boldsymbol{N} = \bar{\boldsymbol{T}}(\boldsymbol{X}, t) \text{ on } \partial\mathcal{B}^{\boldsymbol{T}} \quad , \quad \nabla\alpha \cdot \boldsymbol{N} = 0 \text{ on } \partial\mathcal{B}^{\nabla\alpha} \quad , \quad \nabla d \cdot \boldsymbol{N} = 0 \text{ on } \partial\mathcal{B}^{\nabla d} \tag{17}$$

For a more detailed description of the full model see [3].

Numerical Examples

A characteristic failure mode of porous plastic materials is the so called *cup-cone* failure mechanism, where the plastic deformation leads to a *cup–cone* fracture surface. To demonstrate the capabilities of the presented model a three dimensional cylindrical bar under tension was simulated. Figure 1 shows the load-displacement curve and the influence of the critical work density w_c and the fracture slope parameter ζ on it.

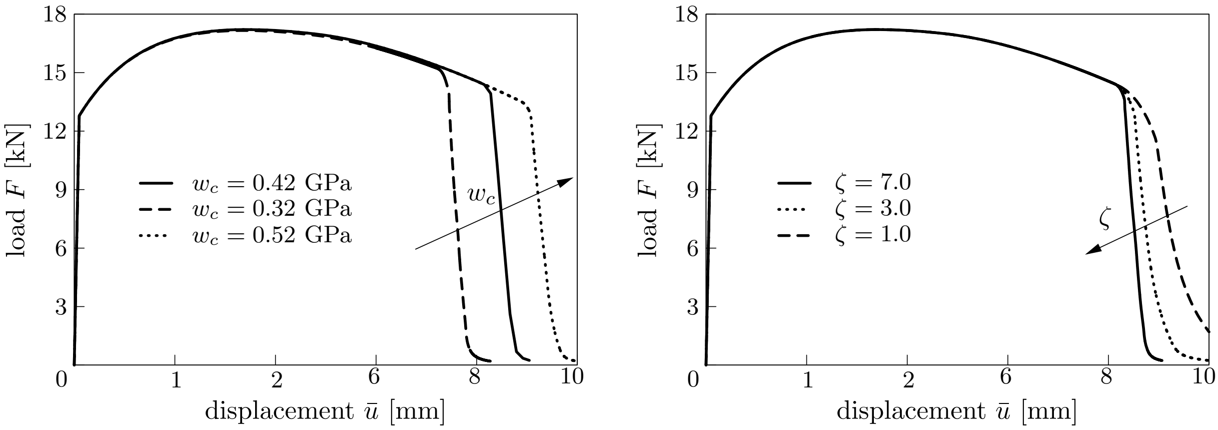

Figure 1. Influence of the fracture parameters w_c and ζ on the crack initiation and propagation.

The deformation and the developing crack surface of the cylindrical bar at different sates are visualized in Figure 2. When the yield limit is reached the bar starts to deform plastically leading to necking. After the onset of plasticity the void volume fraction starts to evolve resulting in a decreasing force. As soon as the threshold of fracture is overcome, e.g. $w^e + w^p > w_c$, the crack starts to propagate from the center on outward to from the *cup-cone* fracture surface and the force vanishes totally.

Conclusions

A variational-based framework for the phase field modeling of fracture in isotropic porous solids was proposed. It incorporates a gradient extended model for porous plasticity and the phase field approach to fracture in context of the finite strain theory. The formulation includes two independent length scale parameters which regularize both the plastic response as well as the crack discontinuities. It was shown that the model is able to represent the characteristic failure mode, i.e. the *cup-cone* failure, of porous plastic material.

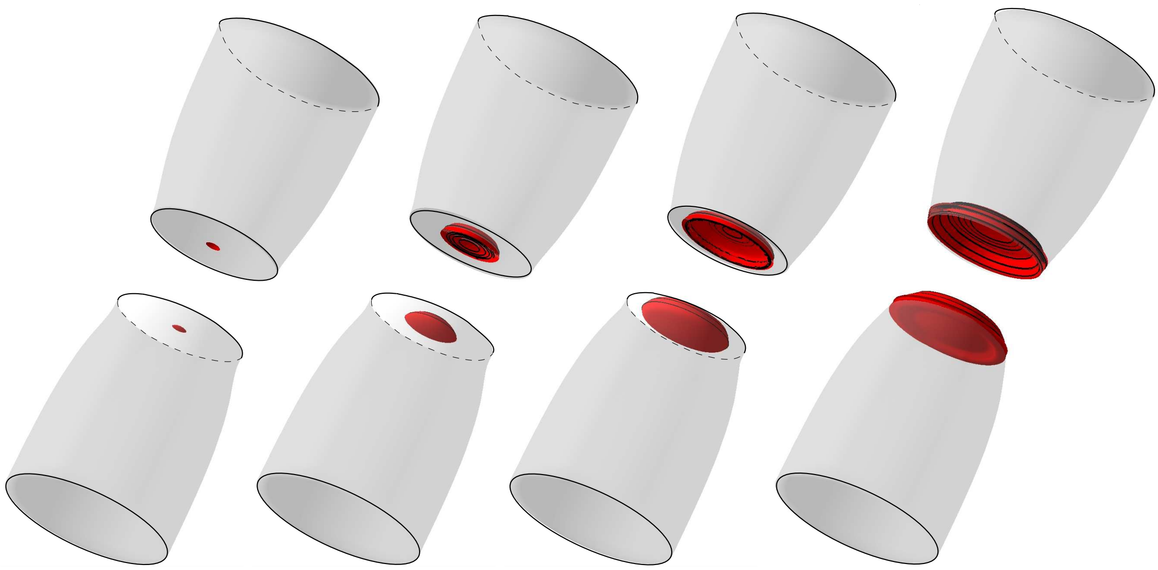

Figure 2. Three dimensional necking of a cylindrical bar. Evolution of the cup–cone failure mechanism.

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