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Recent advances in isogeometric dual mortar patch coupling

Wolfgang Dornisch[1]*, Joachim Stöckler[2] and Ralf Müller[1]

Micro Abstract
Isogeometric analysis fosters the integration of design and analysis by using the geometry description of the CAD system also for the numerical analysis. Hereby, the use of NURBS surfaces is common but entails the need for a coupling of non-conforming patches. The use of mortar methods allows a coupling which requires neither additional variables nor empirical parameters. In this contribution dual basis functions are used in order to obtain an accurate and efficient mortar method.

[1]Chair of Applied Mechanics, University of Kaiserslautern, Kaiserslautern, Germany
[2]Institute of Applied Mathematics, TU Dortmund University, Dortmund, Germany
***Corresponding author**: dornisch@rhrk.uni-kl.de

1 Introduction

A tighter integration between design and analysis is fostered by using the same basis functions for both processes. This idea, which is commonly referred to as isogeometric analysis, has been proposed in [11]. A huge share of works in this field use Non-Uniform Rational B-splines (NURBS), which are also the prevailing standard in the Computer-Aided Design (CAD) industry. NURBS surfaces are able to exactly represent the most common geometrical shapes and their widespread use has led to a multitude of open-source routines which abet a straight-forward integration of NURBS basis functions into existing finite element analysis (FEA) frameworks. NURBS surfaces are defined by a tensor product structure with rectangular parametric spaces. Thus, general structures cannot be described by a single patch but are defined by a multitude of patches which are in general non-conforming along their interfaces. Methods for the handling of non-conforming meshes are required in order to avoid unnecessary mesh refinement. Several methods which enrich the variational formulation have been proposed, e.g. [1, 4, 5, 8]. All these methods either lack universal applicability, robustness or efficiency. This hinders the use of NURBS-based isogeometric analysis in practical applications. The use of mortar methods with constrained approximation spaces [2, 3, 10] or similar constraint formulations [7] is promising due to its universal applicability and robustness. However, the use of standard mortar formulations is inefficient due to an arising global support of nodes along the interfaces. In order to overcome this issue, the use of dual basis functions and approximate dual basis functions has been proposed in [9, 13]. Different approaches for dual basis functions for NURBS are discussed in this work.

2 Isogeometric analysis

The main difference between isogeometric analysis and conventional finite element analysis is the basis functions. NURBS basis functions are computed from B-spline basis functions by introducing an additional weight factor for each control point $i = 1, \dots, n$. B-spline basis functions $N_i^p(\xi)$ are defined by their order p and a knot vector $\boldsymbol{\Xi} = [\xi_i]$ with $i = 1, \dots, n + p + 1$. They can be computed using the Cox–de Boor algorithm (see e.g. [11]). The support of B-spline basis functions is confined to the interval $[\xi_i, \xi_{i+p+1})$. The control points $\boldsymbol{B}_i = \left[\boldsymbol{X}_i^T, w_i\right]^T$ with

$i = 1, \ldots, n$ are used to interpolate the physical location of a NURBS curve by

$$\boldsymbol{X}^h = \sum_{I=1}^{n_{en}} N_I \boldsymbol{X}_I \quad \text{with} \quad N_I(\xi) := \frac{N_i^p(\xi) w_i}{\sum_{\hat{i}=1}^{n_{en}} N_{\hat{i}}^p(\xi) w_{\hat{i}}} \quad \text{and} \quad n_{en} = p + 1 \,. \tag{1}$$

Multidimensional extensions for surfaces and volumes can be found e.g. in [11]. The NURBS basis functions N_I are used as basis functions for the displacements $\boldsymbol{u}^h = \sum_{I=1}^{n_{en}} N_I \boldsymbol{u}_I$ and the virtual test functions $\delta\boldsymbol{u}^h = \sum_{I=1}^{n_{en}} N_I \delta\boldsymbol{u}_I$.

3 Dual basis functions and related concepts

The properties of dual basis functions λ_i are defined by the functional $f_i(N_j) := \int_{\Xi} N_j \lambda_i \, \mathrm{d}s = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Chui et al. [6] introduced the notion of approximate dual basis functions which fulfill the Kronecker delta property only approximately by $f_i^{approx}(N_j) \approx \delta_{ij}$. Dual basis functions for NURBS can amongst others be computed from the inverse of the global Gram matrix, from a local Gram matrix [13] or explicitly using polynomials and a jump function [12]. The ability to exactly reproduce polynomials of degree $r = 0, \ldots, p$ by

$$x^r = \sum_{j=1}^{n} c_j^r \lambda_j \quad \text{with} \quad c_j^r = \int_I x^r N_j \, \mathrm{d}s \tag{2}$$

is a prerequisite for optimal convergence of the mortar method if dual basis functions λ_i are used as test function; see [9] for more details. The reproduction property stated in Eq. (2) is fulfilled by the global Gram dual basis functions and by the approximate dual basis functions, but not by the the approaches proposed in [12, 13]. In terms of support the situation is different. The local Gram duals and the explicitly defined duals have support in $p + 1$ elements, the same as the NURBS basis functions. The approximate dual basis functions have support in $3p + 1$ elements and the global Gram dual basis functions have support in all n interface elements. More details and justifications for this can be found in [9]. For a better comparison all four approaches are compared in Tab. 1.

4 Mortar-based patch coupling applied to linear elasticity

In this work the mortar method with constrained approximation spaces [2] is used due to its universal applicability. In this method neither the variational formulation is altered nor empirical parameters or additional unknowns are required. It can be used without alterations for more complicated settings and the resulting stiffness matrix is positive definite. Linear elasticity is chosen here as a proof of concept for the sake of compactness. The application of this method in an isogeometric framework in order to couple non-conforming NURBS patches has been proposed in [10].

The domain Ω is decomposed into complementary subdomains Ω^i with $i = 1, \ldots, n_\mathrm{p}$. Without loss of generality we restrict ourselves to a two patch setting ($n_\mathrm{p} = 2$) with a classification into

Approach	**Proposed in**	**fulfills Eq. (2)**	**support**	**efficiency**	**accuracy**
Global Gram dual	[2, 9, 10]	yes	n	$--$	$++$
Local Gram dual	[13]	no	$p+1$	$++$	$--$
Explicit dual	[12]	no	$p+1$	$++$	$--$
Approximate dual	[6, 9]	yes	$3p+1$	$+$	$+$

Table 1. Comparison of different approaches for the computation of dual basis functions

a master patch (ma) and a slave patch (sl), where $\Omega = \Omega^{\mathrm{ma}} \cup \Omega^{\mathrm{sl}}$ and $\Gamma_{\mathrm{c}} = \Omega^{\mathrm{ma}} \cap \Omega^{\mathrm{sl}}$ holds. Equilibrium of forces, Neumann and Dirichlet boundary conditions are defined by

$$\mathrm{Div}\, \boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0} \quad \text{in } \Omega\,, \quad \boldsymbol{t} = \bar{\boldsymbol{t}} \quad \text{on } \Gamma_{\mathrm{N}} = \Gamma_{\mathrm{N}}^{\mathrm{ma}} \cup \Gamma_{\mathrm{N}}^{\mathrm{sl}} \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{D}} = \Gamma_{\mathrm{D}}^{\mathrm{ma}} \cup \Gamma_{\mathrm{D}}^{\mathrm{sl}}\,, \tag{3}$$

respectively. The interface condition $\boldsymbol{u}^{\mathrm{ma}} = \boldsymbol{u}^{\mathrm{sl}}$ on Γ_{c} is not fulfilled in a strong way, but by choosing the constrained solution space

$$\mathcal{S} := \left\{ \boldsymbol{u} \in \left(H^1(\Omega)\right)^2 \,\middle|\, \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{D}}\,, \int_{\Gamma_{\mathrm{c}}} \left(\boldsymbol{u}^{\mathrm{sl}} - \boldsymbol{u}^{\mathrm{ma}}\right) \cdot \boldsymbol{\lambda}\, \mathrm{d}s = 0 \quad \forall\, \boldsymbol{\lambda} \in \left(H^{\frac{1}{2}} \left(\Gamma_{\mathrm{c}}^{\mathrm{sl}} \right)' \right)^2 \right\} \tag{4}$$

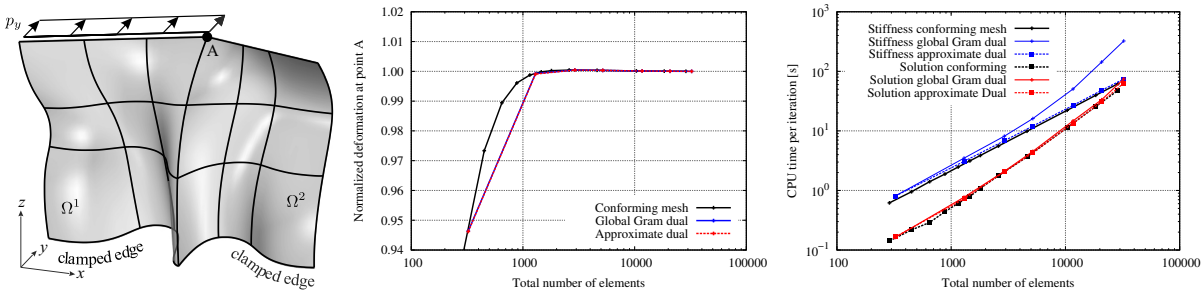
which enforces the interface condition in a weak manner. The weak form of the boundary value problem stated in Eq. (3) is referred to as $G(\boldsymbol{u}, \delta\boldsymbol{u})$. It is solved by the value $\boldsymbol{u} \in \mathcal{S}$ which fulfills

$$G(\boldsymbol{u}, \delta\boldsymbol{u}) = 0 \quad \forall \delta\boldsymbol{u} \in \mathcal{S}\,. \tag{5}$$

The discretization using the finite element method and details on the computation of the dual basis functions can be found in [9].

5 Numerical example

Studies of the convergence behavior of the global Gram duals, the explicit duals and the approximate duals can be found in [9], whereas for results for the local Gram duals we refer to [13]. In both works the convergence of the global stress error of computations of non-conforming discretizations of the elastic plate with hole is assessed, whereby different kinds of test functions are used for the mortar method. In [9] it is shown, that the global Gram duals and the approximate duals yield nearly optimal convergence behavior, while the explicit duals introduce a considerable interface error which degrades the global stress convergence behavior. In [13] is is shown that also the local Gram duals introduce a considerable interface error. Thus, in the following only the approximate duals and the global Gram duals are considered and the focus is set on efficiency. This is assessed with the help of a curved shell structure. Computations are performed with conforming meshes as shown in Fig. 1a, and with non-conforming meshes with a refinement ration of 15 : 12 along the interface, which is the straight line between the point A and the lower edge. The deformation convergence behavior of point A is shown in Fig. 1b and here all computations agree very well. The related computational costs are given in Fig. 1c. The computational costs for the solution of the global system of equations are very similar for the two non-conforming approaches and only slightly higher than for the conforming mesh. The situation is different for the formation of the stiffness matrix. Here the approximate duals yield only slightly higher costs than the conforming discretization. But due to their global support, the global Gram duals yield several times higher costs than the conforming discretization. This shows that out of the four considered concepts in Tab. 1 only the approximate duals yield both accurate and efficient computations.



(a) System sketch **(b)** Deformation convergence **(c)** Computational costs

Figure 1. Comparison of deformation convergence behavior and computational costs between the global Gram dual basis functions and the approximate dual basis functions for computations of non-conforming meshes with $15j : 12j$-refinement along the interface

6 Conclusion

This work gives an overview on different concepts for the computation of dual basis functions for NURBS and the implications on the efficiency and accuracy of the isogeometric mortar method. The numerical results show that the approximate dual basis functions are the best compromise between accuracy and efficiency.

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