Proceedings of the 7th GACM Colloquium on Computational Mechanics
for Young Scientists from Academia and Industry
October 11-13, 2017 in Stuttgart, Germany

A Phase Field Approach to Fracture in Anisotropic Brittle Solids

Stephan Teichtmeister[1]*, Daniel Kienle[1], Fadi Aldakheel[2] and Marc-André Keip[1]

Micro Abstract
A phase field model of fracture that accounts for anisotropic material behavior and crack propagation is presented within the small and large deformation context. Different kinds of material anisotropy are incorporated by (i) enhancing the crack surface density function by appropriate structural tensors stemming from a rigorous application of the theory of tensor invariants and (ii) by a modification of energetic and stress-like fracture criteria.

[1]Institute of Applied Mechanics (CE), University of Stuttgart, Stuttgart, Germany
[2]Institute of Continuum Mechanics, Leibniz Universität Hannover, Hannover, Germany
***Corresponding author**: stephan.teichtmeister@mechbau.uni-stuttgart.de

Phase Field Approximation of Anisotropic Crack Topology

Consider a bounded domain $\mathcal{B} \subset \mathcal{R}^\omega$, $\omega \in \{2,3\}$ with boundary $\partial\mathcal{B}$. We introduce the crack phase field $d\colon \mathcal{B} \times \mathcal{T} \to [0,1]$, $(\boldsymbol{x},t) \mapsto d(\boldsymbol{x},t)$, where $d=0$ characterizes the unbroken and $d=1$ the fully broken state of the material at $\boldsymbol{x} \in \mathcal{B}$. The parameter t represents for rate-independent problems an incremental loading parameter and for rate-dependent problems the time. The regularization of a sharp crack topology in *isotropic solids* bases on the crack surface density function

$$\gamma_l(d,\nabla d) = \frac{1}{2l}d^2 + \frac{l}{2}\nabla d\cdot\nabla d \quad \text{and} \quad \gamma_l(d,\nabla d,\nabla^2 d) = \frac{1}{2l}d^2 + \frac{l}{4}\nabla d\cdot\nabla d + \frac{l^3}{32}\nabla^2 d : \nabla^2 d\,, \quad (1)$$

respectively, in terms of the fracture length scale parameter l, see [4] and [1]. To outline an extension to a class of *anisotropic response*, we consider the anisotropic crack surface density function up to second order satisfying

$$\gamma_l(d,\boldsymbol{Q}\star\nabla d,\boldsymbol{Q}\star\nabla^2 d) = \gamma_l(d,\nabla d,\nabla^2 d) \quad \text{for all } \boldsymbol{Q}\in\mathcal{G}\subset\mathcal{O}(3) \quad (2)$$

with $\mathcal{G}$ denoting the symmetry group of the given anisotropic material. Following the representation theory of isotropic tensor functions outlined e.g. in [8], a quadratic function must have the form

$$\gamma_l(d,\nabla d,\nabla^2 d;\boldsymbol{A},\mathbb{A}) = \gamma_l(d^2,\nabla d\cdot\nabla d,\nabla d\cdot\boldsymbol{A}\,\nabla d,\nabla^2 d:\nabla^2 d,\nabla^2 d:\mathbb{A}:\nabla^2 d) \quad (3)$$

in terms of the symmetric second- and fourth-order structural tensors $\boldsymbol{A}$ and $\mathbb{A}$. For an orthotropic microstructure based on three structural directors $\{\boldsymbol{a}_i\}_{i=1,3}$ satisfying $\|\boldsymbol{a}_i\| = 1$, $\boldsymbol{a}_i\cdot\boldsymbol{a}_j = \delta_{ij}$ and $\boldsymbol{a}_i\times\boldsymbol{a}_j = \epsilon_{ijk}\boldsymbol{a}_k$ we can for instance introduce the simple structural tensors

$$\boldsymbol{A} = \boldsymbol{1} + \alpha^1\boldsymbol{M}_1 + \alpha^2\boldsymbol{M}_2 \quad \text{and} \quad \mathbb{A} = \mathbb{I}_{\boldsymbol{A}} = \frac{1}{2}(\,[\boldsymbol{A}]_{ik}[\boldsymbol{A}]_{jl} + [\boldsymbol{A}]_{il}[\boldsymbol{A}]_{jk}\,)\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l \quad (4)$$

in terms of $\boldsymbol{M}_i = \boldsymbol{a}_i\otimes\boldsymbol{a}_i$ and two material parameters α^1,α^2 only[1]. More complicated forms are given in [7]. For a cubic microstructure a simple fourth-order structural tensor depending on

[1]For a simple treatment of transverse isotropy characterized by the structural vector $\boldsymbol{a} = \boldsymbol{a}_1$ representing the fiber orientation, we just set $\alpha^2 = 0$ in (4) and $(6)_1$.

two material parameters α, β has the form

$$\mathbb{A} = \mathbb{I} + \alpha(\boldsymbol{M}_1 \otimes \boldsymbol{M}_1 + \boldsymbol{M}_2 \otimes \boldsymbol{M}_2) + \beta\,\mathrm{sym}(\boldsymbol{M}_1 \otimes \boldsymbol{M}_2)\,. \tag{5}$$

Note, that the material parameters have to lie within the open ranges $(\alpha^1, \alpha^2) = (-1, \infty) \times (-1, \infty)$ and $\alpha > -1$ along with $|\beta| < 2|1 + \alpha|$, respectively, to ensure positive definiteness of the corresponding structural tensors defined in (4) and (5). Finally, we consider a point $\boldsymbol{x}_\Gamma$ on a straight sharp crack $\Gamma \subset \mathcal{B} \subset \mathcal{R}^2$ inclined under an angle φ, which is located sufficiently far away from the crack tip. Assuming the (effective) fracture length scale to be small compared to the length $|\Gamma|$ of the sharp crack, the structural tensors $(4)_1$ and (5) of orthotropy and cubic symmetry imply the effective length scale parameters

$$l^* = l[1 + \alpha^1 \sin^2(\varphi - \theta) + \alpha^2 \cos^2(\varphi - \theta)] \quad \text{and} \quad l^{**} = \gamma\{1 + \delta \cos[4(\varphi - \theta)]\}^{\frac{1}{3}} \tag{6}$$

in terms of the given angle θ representing the inclination of the structural director $\boldsymbol{a}_1$, and the material parameters α^1, α^2 and γ, δ, respectively, latter depending on l, α, β. Note, that the cubic length scale parameter $(6)_2$ shows a classical four-fold symmetry, whereas the orthotropic length scale parameter $(6)_1$ yields a two-fold symmetry.

Evolution Problem of Anisotropic Phase Field Fracture

In the small-strain setting, the motion of the fracturing solid body is described by the displacement field $\boldsymbol{u}\colon \mathcal{B} \times \mathcal{T} \to \mathcal{R}^\omega$, $(\boldsymbol{x}, t) \mapsto \boldsymbol{u}(\boldsymbol{x}, t)$. We consider a "total" pseudo energy functional and a dissipation potential functional

$$W(\boldsymbol{u}, d; \boldsymbol{A}, \mathbb{A}) = \int_{\mathcal{B}} w(\boldsymbol{\varepsilon}, d, \nabla d, \nabla^2 d; \boldsymbol{A}, \mathbb{A})\, dV \quad \text{and} \quad V(\dot{d}) = \int_{\mathcal{B}} v(\dot{d})\, dV \tag{7}$$

in terms of a "total" pseudo energy density function w and a convex, non-smooth dissipation potential density function v. With these two functionals at hand, the variational principle for the evolution problem of phase field fracture reads

$$\{\dot{\boldsymbol{u}}, \dot{d}\} = \mathrm{Arg}\{\inf_{\dot{\boldsymbol{u}} \in \mathcal{W}_{\dot{\boldsymbol{u}}}} \inf_{\dot{d} \in H^2} [\frac{d}{dt} W + V]\}\,, \tag{8}$$

with $\mathcal{W}_{\dot{\boldsymbol{u}}} = \{\dot{\boldsymbol{u}} \in H^1(\mathcal{B}) |\, \dot{\boldsymbol{u}} = \dot{\bar{\boldsymbol{u}}} \text{ on } \partial\mathcal{B}^{\boldsymbol{u}}\}$ according to the decomposition of the boundary into Dirichlet and Neumann parts. The Euler equations of the variational principle (8) are simply the quasi-static stress equilibrium and the crack phase field evolution equation

$$\delta_{\boldsymbol{u}} w = \boldsymbol{0} \quad \text{and} \quad 0 \in \delta_d w + \partial_{\dot{d}} v \tag{9}$$

along with the Neumann-type boundary conditions. Accounting for the irreversibility $\dot{d} \geq 0$ of the fracture phase field, the dissipation potential function reads $v(\dot{d}) = I(\dot{d}) + \frac{\eta}{2}\dot{d}^2$ in terms of the indicator function of positive real numbers and a mobility parameter η. With this specific form, the evolution equation for the crack phase field attains a *Ginzburg-Landau-type* structure which, by a specific choice of the degradation function, may be recast into

$$\widetilde{\eta}\dot{d} = (1 - d)\mathcal{H} + l\,\delta_d \gamma_l \quad \text{with} \quad \mathcal{H} = \max_{s \in [0,T]} \widetilde{D}(state(\boldsymbol{x}, s)) \tag{10}$$

in terms of the crack driving state function $\widetilde{D}$, see [3]. As a specific example, we choose the "total" pseudo energy density function

$$w(d, \nabla d, \nabla^2 d; \boldsymbol{A}, \mathbb{A}) = (1 - d)^2 \widetilde{\psi}(\boldsymbol{\varepsilon}) + g_c \gamma_l(d, \nabla d, \nabla^2 d; \boldsymbol{A}, \mathbb{A}) \tag{11}$$

in terms of the (anisotropic) effective energy stored per unit volume in the undamaged bulk and the material parameter g_c which within this setting does *not* represent Griffith's critical

energy release rate. The representation (11) characterizes a brittle fracture model without an elastic phase and yields the crack driving state function $\widetilde{D} = 2\widetilde{\psi}(\boldsymbol{\varepsilon})/(g_c/l)$. To distinguish between energetic tensile and compressive parts within the anisotropic material, the effective stress $\widetilde{\boldsymbol{\sigma}} = \partial_{\boldsymbol{\varepsilon}}\widetilde{\psi}$ is decomposed into a positive part $\widetilde{\boldsymbol{\sigma}}^+$ and a negative part $\widetilde{\boldsymbol{\sigma}}^-$ via a spectral representation of $\widetilde{\boldsymbol{\sigma}}$. For e.g. a transversely isotropic material with the fiber direction represented by the structural vector $\boldsymbol{a}$, the positive and negative stress function follow as

$$\widetilde{\psi}^{*\pm}(\widetilde{\boldsymbol{\sigma}}) = p_1\langle \mathrm{tr}[\widetilde{\boldsymbol{\sigma}}]\rangle_\pm^2 + p_2\,\mathrm{tr}[\widetilde{\boldsymbol{\sigma}}^\pm\widetilde{\boldsymbol{\sigma}}^\pm] + p_3\langle \mathrm{tr}[\widetilde{\boldsymbol{\sigma}}(\boldsymbol{a}\otimes\boldsymbol{a})]\rangle_\pm^2 \tag{12}$$

in terms of the ramp functions of $\mathcal{R}_+$ and $\mathcal{R}_-$ expressed by the Macaulay bracket. With a decomposition of the form (12) at hand, we modify the crack driving state function given above in the way $\widetilde{D} = 2\widetilde{\psi}^{*+}(\widetilde{\boldsymbol{\sigma}})/(g_c/l)$.

Anisotropic Fracture Toughness and Crack Propagation

To calculate the accumulated dissipation $\mathcal{D}$ per unit crack length, we again consider a straight crack $\Gamma \subset \mathcal{B} \subset \mathcal{R}^2$ for simplicity and express the crack phase field in terms of the coordinate η, the coordinate line of which is perpendicular to the crack with $\eta = 0$ locating a point on Γ. For an orthotropic material characterized by the second-order structural tensor $(4)_1$ we obtain

$$\mathcal{D} = g_c\int_{-\infty}^{+\infty}\left[\frac{1}{2l}d(\eta)^2 + \frac{l^*}{2}d'(\eta)^2\right]d\eta = g_c\sqrt{\frac{l^*}{l}}\,. \tag{13}$$

Then, as long as the effective length scale is small compared to the crack length and the dimension of the body, the accumulated dissipation due to fracture plays the role of Griffith's crititcal energy release rate

$$G_c(\varphi;\theta) = \mathcal{D}\,, \tag{14}$$

see [5] for the case of isotropy. Apparently the fracture toughness depends on the angle of crack propagation relatively to a given structural director. Note, that in the isotropic scenario we obtain $G_c = g_c$. According to [2] the crack propagation angle φ under quasi-static loading is the one for which Griffith's criterion is first reached such that

$$\frac{G(\varphi)}{G_c(\varphi;\theta)} \quad \text{is maximized globally,} \tag{15}$$

where $G(\varphi)$ denotes the energy release rate. At this point the distinction between weakly and strongly anisotropic systems should be mentioned, the latter one allows the crack to be guided along forbidden directions, see [6] and [7].

Numerical Example

We consider as a boundary value problem a square plate which contains a notch running from the left edge to the center of the body. The bottom of the specimen is fixed in vertical direction whereas at the top a linear increasing displacement in vertical direction is applied. The material is transversely isotropic with fiber direction given by the structural director $\boldsymbol{a}$ which is inclined under 45°. The anisotropic crack topology is modeled by the second-order structural tensor $(4)_1$ with $\alpha^2 = 0$. The crack path can theoretically be predicted by the criterion (15) using expression (13) for the fracture toughness (weakly anisotropic system). Following [6], latter can graphically be translated in the following way: the crack angle is determined by the point on the $G_c^{-1}(\varphi;\theta)$ polar plot which is first tangentially touched by a vertical line moving continuously from right to left during monotonous loading of the specimen, see Figure 1, where the mentioned tangency point is marked in red. The leftwards moving vertical line represents the polar plot of the reciprocal energy release rate $G^{-1}(\varphi)$, what is only an approximation here. This, and the circumstance, that the sensitivity of the crack propagation angle φ with respect to the anisotropy

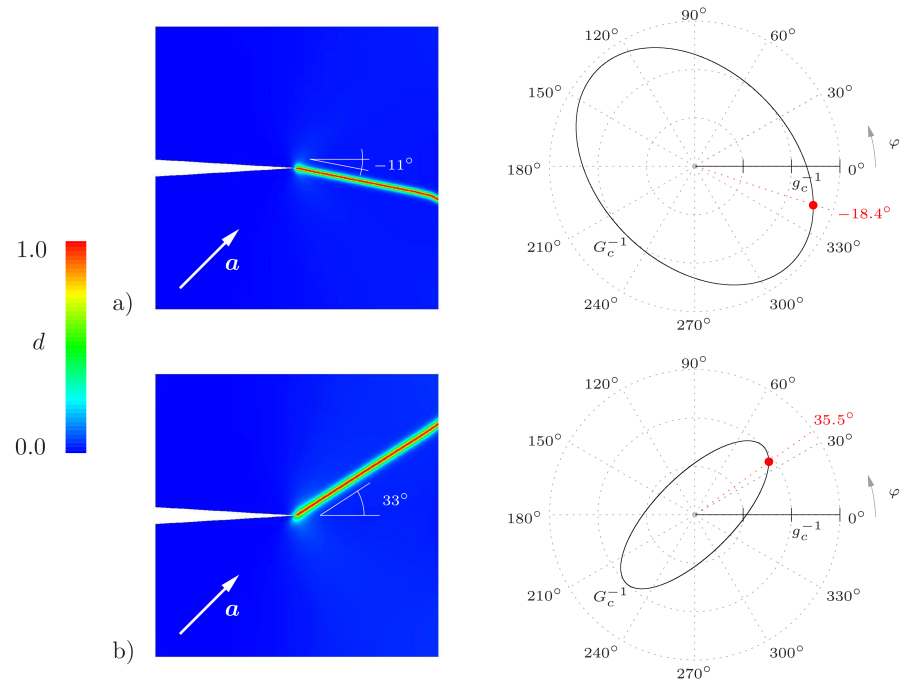

Figure 1. Fracture of a transversely isotropic specimen under tension for a) $\alpha^1 = -0.5$ and b) $\alpha^1 = 5.0$.

parameter α^1 is high in the low-α^1 range, lead for small α^1 to a deviation of the theoretically predicted crack propagation angle from the one obtained by the phase field simulation. For a higher anisotropy parameter however, the theoretical predictions of φ and the numerical results are in good agreement, see Figure 1. Apparently the larger α^1 is, the later the specimen cracks.

References

[1] M. J. Borden, T. J. R. Hughes, C. M. Landis, and C. V. Verhoosel. A higher-order phase-field model for brittle fracture: Formulation and analysis within the isogeometric analysis framework. *Computer Methods in Applied Mechanics and Engineering*, 273:100–118, 2014.

[2] A. Chambolle, G. Francfort, and J.-J. Marigo. When and how do cracks propagate? *Journal of the Mechanics and Physics of Solids*, 57:1614–1622, 2009.

[3] C. Miehe, L. Schänzel, and H. Ulmer. Phase field modeling of fracture in multi-physics problems. Part I. Balance of crack surface and failure criteria for brittle crack propagation in thermo-elastic solids. *Computer Methods in Applied Mechanics and Engineering*, 294:449–485, 2015.

[4] C. Miehe, F. Welschinger, and M. Hofacker. Thermodynamically consistent phase-field models of fracture: Variational principles and multi-field FE implementations. *International Journal of Numerical Methods in Engineering*, 83:1273–1311, 2010.

[5] P. Sicsic and J.-J. Marigo. From gradient damage laws to Griffith's theory of crack propagation. *Journal of Elasticity*, 113:55–74, 2013.

[6] A. Takei, B. Roman, J. Bico, E. Hamm, and F. Melo. Forbidden directions for the fracture of thin anisotropic sheets: An analogy with the Wulff plot. *Physical Review Letters*, 110:144301–1–5, 2013.

[7] S. Teichtmeister, D. Kienle, F. Aldakheel, and M.-A. Keip. Phase field modeling of fracture in anisotropic brittle solids. *International Journal of Non-Linear Mechanics*, in press, 2017.

[8] Q. S. Zheng. Theory of representations for tensor functions – a unified invariant approach to constitutive equations. *Applied Mechanical Review*, 47(11):546–587, 1994.