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Hierarchic Isogeometric Large Rotation Shell Elements Including Linearized Transverse Shear Parametrization

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Micro Abstract
Two novel hierarchic isogeometric formulations for geometrically nonlinear shell analysis including transverse shear effects are presented. Both concepts combine a fully nonlinear rotation-free Kirchhoff-Love shell model with hierarchically added linearized transverse shear components. An additive split of Green-Lagrange strains dramatically facilitates representing large rotations in shell analysis while the proposed hierarchic concepts are intrinsically free from transverse shear locking.

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Introduction

The development of shell formulations with transverse shear effects is a broadly investigated topic for the last decades and also in the present there are still some open questions. The development of the isogeometric concept by Hughes and co-workers [6] enables an easy construction of C^1-continuous approximation spaces with NURBS (Non-Uniform Rational B-Splines) shape functions. In the contribution Kiendl et al. [7] this feature was used to formulate an isogeometric rotation-free Kirchhoff-Love type shell, where large rotations can be easily handled. For thicker shells, shear deformable Reissner-Mindlin type isogeometric shell elements have been proposed by [1–4]. Compared to rotation-free thin shells, handling large rotations is more complicated and (additional) transverse shear locking is introduced. This contribution presents two novel Reissner-Mindlin type shell formulations inspired by hierarchic parameterizations [5, 8]. Transverse shear effects are hierarchically included in an additive way, dramatically facilitating the representation of large rotations, while being intrinsically free from transverse shear locking [9, 10].

1 Hierarchic concept for a Timoshenko beam formulation

The hierarchic concept and the reason for shear locking can be explained in a picturesque way on the example of a two-dimensional Timoshenko beam, which is based on a Bernoulli model with hierarchically added shear components. Standard Timoshenko beam formulations use the vertical displacement v and the total rotation of the cross section φ as primal variables, leading to the following kinematic equations, where the curvature is denoted by κ and the shear angle by γ:

$$\kappa = \varphi', \qquad \gamma = v' + \varphi. \tag{1}$$

The shear angle γ is a combination of the angle due to the vertical displacement and the total rotation φ of the cross section. In the internal forces for this model problem locking can be observed. With increasing slenderness L/t the approximation of the bending moment deteriorates and shear forces exhibit strong oscillations. The reason for this locking behavior lies in the unbalance of function spaces within the kinematic equation of the shear angle γ in the case of equal order interpolation. Using quadratic shape functions, the sum of a linear and a quadratic function can not lead to a constant shear angle of zero, which is required to represent pure bending. Following the hierarchic concept of [5], Timoshenko kinematics can also be formulated with the Bernoulli beam theory as base and additional shear components. One possibility is to set the vertical displacement v and the shear angle γ as primal variables, leading to the kinematic equations

$$\kappa = -v'' + \gamma', \qquad \gamma = \gamma. \tag{2}$$

In those equations the shear angle is now decoupled from the midsurface displacements and therefore, pure bending can be represented and shear locking is eliminated. As the unbalance of the derivatives moved to the curvature, oscillations still occur in the shear forces. Second derivatives of the midsurface now appear in the curvature term, like in the Bernoulli beam formulation. This results in a necessity of an inter-element C^1-continuity. Under the same conditions, a full balance of the kinematic equations can be achieved by a split of the total displacement $v = v_b + v_s$ into two components due to shear v_s and bending v_b. For an easier treatment of boundary conditions, v and v_s are used as primal variables. The shear angle is now included as a derivative of the "shear displacement". In the resulting kinematic equations

$$\kappa = -v'' + v_s'', \qquad \gamma = v_s', \tag{3}$$

the same additive split of Bernoulli beam solution and additive shear component can be identified. This split removes the unbalance in the kinematics and therefore all phenomena of shear locking, namely the underestimation of displacements and oscillations in the shear forces are avoided while retaining equal order interpolation.

2 Hierarchic concept for a Reissner-Mindlin shell

The hierarchic concept can be transferred to shell formulations. The Bernoulli beam kinematics is equivalent to a Kirchhoff-Love (KL) shell with the same C^1-continuity requirement, whereas a Reissner-Mindlin (RM) shell includes transverse shear effects. Those effects can be parameterized by rotational degrees of freedom where a C^0-continuity is sufficient.

For the shell formulation, two configurations are defined: the reference and the current configuration with the position vectors $\mathbf{X}$ and $\mathbf{x}$ on a point in the shell body and $\mathbf{R}$ and $\mathbf{r}$ on a point in the shell midsurface. Upper case letters indicate the reference configuration and lower case letters the current configuration, see Figure 1. The covariant base vectors of the midsurface are denoted by $\mathbf{A}_i$ and $\mathbf{a}_i$, respectively. The two configurations are connected by the displacement vectors $\mathbf{u}$ of the shell body and $\mathbf{v}$ of the shell midsurface.

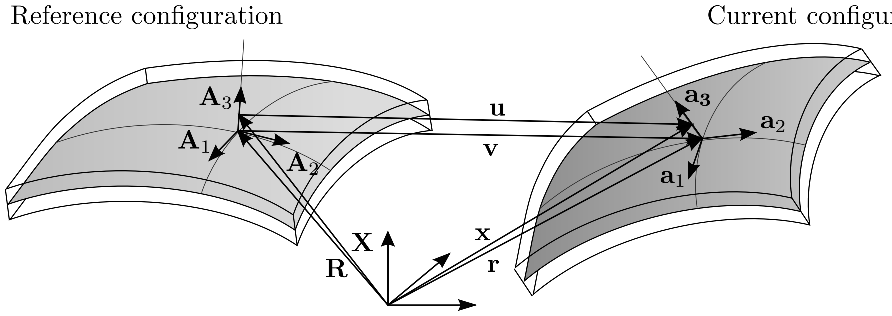

Figure 1. Reference and current configuration of a shell

2.1 Small rotations

The standard formulation of a Reissner-Mindlin shell model incorporates transverse shear effects by a rotation of the director by the total rotation including the shear rotations. For a hierarchic formulation of the Reissner-Mindlin shell model, a Kirchhoff-Love shell formulation is taken as a basis, see Figure 2 (left). In the latter, the director is rotated by the linearized rotation vector Φ, which only depends on the midsurface displacements $\mathbf{v}$. To obtain a Reissner-Mindlin formulation, a hierarchic difference vector $\mathbf{w}$, which considers transverse shear effects, is superimposed to the rotated Kirchhoff-Love type director $\mathbf{a}_3^{\perp}$. The additive split in the formulation can also be observed in the linearized Green-Lagrange strain components.

$$\varepsilon_{11}^{\mathrm{RM}} = \varepsilon_{11}^{\mathrm{KL}} + \xi^3(\mathbf{w}_{,1} \cdot \mathbf{A}_1) \tag{4}$$

$$2\varepsilon_{12}^{\mathrm{RM}} = \varepsilon_{12}^{\mathrm{KL}} + \xi^3(\mathbf{w}_{,1} \cdot \mathbf{A}_2 + \mathbf{w}_{,2} \cdot \mathbf{A}_1) \tag{5}$$

$$\varepsilon_{22}^{\mathrm{RM}} = \varepsilon_{22}^{\mathrm{KL}} + \xi^3(\mathbf{w}_{,2} \cdot \mathbf{A}_2) \tag{6}$$

$$2\varepsilon_{13}^{\mathrm{RM}} = \mathbf{w} \cdot \mathbf{A}_1 \tag{7}$$

$$2\varepsilon_{23}^{\mathrm{RM}} = \mathbf{w} \cdot \mathbf{A}_2 \tag{8}$$

This decouples the shear terms in the strain components from the midsurface displacements and shear locking is eliminated. For a difference vector of zero ($\mathbf{w} = 0$), the Kirchhoff-Love solution is recovered.

2.2 Large rotations

In the nonlinear case, the treatment of rotations requires further considerations. The standard parameterization with total rotations as a degrees of freedom requires a feasible parameterization of the rotation space ($SO3$). Furthermore, the unbalance in the kinematic equations, and therefore shear locking, is still present. On the other hand, a Kirchhoff-Love formulation does not use any rotational degrees of freedom as the rotation is described by the partial derivatives of the midsurface displacements and the director always stays perpendicular to the current covariant base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. To apply the hierarchic concept to large rotations, a hierarchic difference vector $\mathbf{w}$ is added to the Kirchhoff-Love type director, as illustrated in Figure 2 (middle). The resulting vector still needs to be normalized to fulfill the inextensibility condition.

$$\mathbf{a}_3^{\mathrm{RM}} = \frac{\mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{w}}{\|\mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{w}\|} \tag{9}$$

With this normalization, the additive characteristics of the hierarchic formulation are lost for large rotations. However, typical applications for Reissner-Mindlin shell formulation include large rotations, but the shear parts of the rotations are mostly very small. With this observation, the transverse shear components may be introduced in a linearized fashion, neglecting small elongations of the director $\mathbf{a}_3^{\mathrm{RM}}$, see Figure 2 (right).

$$\mathbf{a}_3^{\mathrm{RM}} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} + \mathbf{w} \tag{10}$$

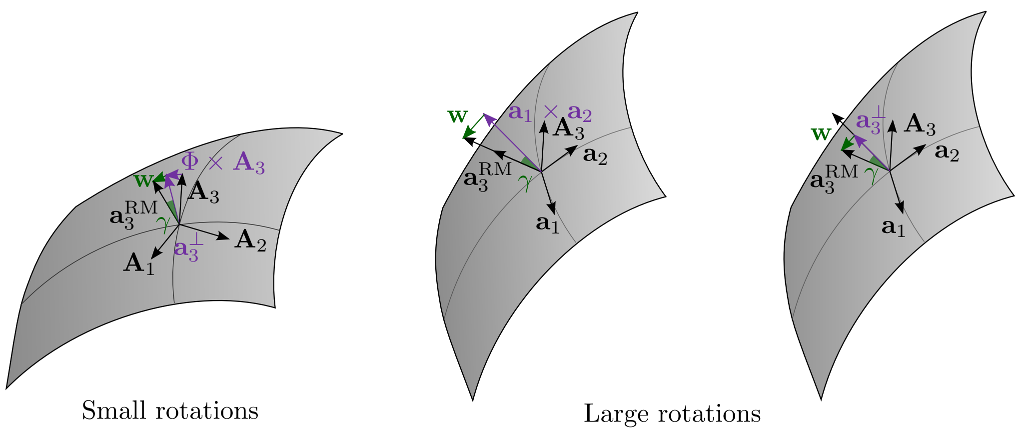

Figure 2. Hierarchic formulation for a Reissner-Mindlin shell for small and large rotations

The linearization of the shear components restores the additive property for the hierarchic Reissner-Mindlin shell, even for the nonlinear case. It also shows a similar structure for the Green-Lagrange strains compared to the linearized strains.

$$E_{11}^{\mathrm{RM}} = E_{11}^{\mathrm{KL}} + \xi^3(\mathbf{w}_{,1} \cdot \mathbf{A}_1) \tag{11}$$
$$2E_{12}^{\mathrm{RM}} = E_{12}^{\mathrm{KL}} + \xi^3(\mathbf{w}_{,1} \cdot \mathbf{A}_2 + \mathbf{w}_{,2} \cdot \mathbf{A}_1) \tag{12}$$
$$E_{22}^{\mathrm{RM}} = E_{22}^{\mathrm{KL}} + \xi^3(\mathbf{w}_{,2} \cdot \mathbf{A}_2) \tag{13}$$
$$2E_{13}^{\mathrm{RM}} = \mathbf{w} \cdot \mathbf{A}_1 \tag{14}$$
$$2E_{23}^{\mathrm{RM}} = \mathbf{w} \cdot \mathbf{A}_2 \tag{15}$$

The framework so far for small and large rotations can be applied to both formulations with hierarchic shear parameterizations, namely hierarchic rotations and hierarchic displacements. They only differ in the components of the hierarchic difference vector $\mathbf{w}$. In the formulation with hierarchic rotations, the hierarchic difference vector

$$\mathbf{w} = w^1\mathbf{a}_1 + w^2\mathbf{a}_2 \tag{16}$$

contains the components w^1 and w^2 that include the shear angles. On the other hand, the parameterization with hierarchic displacements uses the partial derivatives of the shear displacements as components of the hierarchic difference vector

$$\mathbf{w} = v_{,1}^{s1}\mathbf{a}_1 + v_{,2}^{s2}\mathbf{a}_2. \tag{17}$$

The structure of the Green-Lagrange strains and the construction of the current director for linear and nonlinear kinematics is similar for both formulations.

3 Numerical example

A single span beam, see Figure 3, serves as a numerical example to validate the assumption of small shear angles for geometrically nonlinear analyses. It is loaded by a uniform load, which is scaled by the thickness to the power of 3. The reference solution is calculated with the commercial software ANSYS and the shear-deformable shell element SHELL181, which avoids locking by assumed natural strains. The displacement values were calculated with a discretization of 160 elements. For a thick beam ($L/t = 10$), where the transverse shear can not be neglected, the converged values of the maximum vertical displacement between the hierarchic formulations ($u_z = 2.3238$ both) and the reference solution ($u_z = 2.3262$) differ only by approximately 0.1% despite the linearized shear components. In the case of a thin beam ($L/t = 100$), no significant difference can be recognized. Additionally, for the hierarchic displacement formulation, no oscillations occur in the shear forces for the thick as well as for the thin beam.

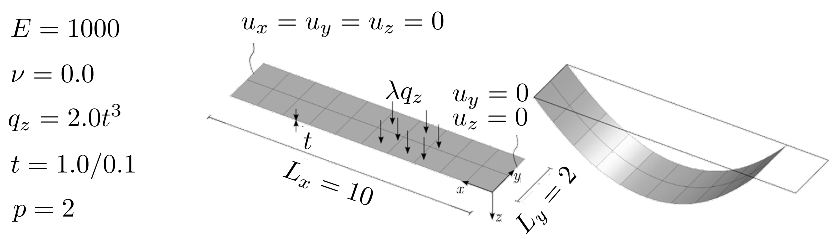

Figure 3. Single span beam problem as a numerical example

Conclusions

In this contribution two different hierarchic finite element formulations for large rotation Reissner-Mindlin shell analysis are presented. Both formulations include hierarchically added linearized transverse shear components although the total rotations may be arbitrarily large. It was shown, that even for a significant thickness, displacement values and internal forces resemble the results from a fully nonlinear formulation, despite the linearization of the shear components. As the unbalance in the kinematic equations is completely removed, the formulations are intrinsically free from shear locking by construction, independent of the discretization. Due to the additive characteristics of the Kirchhoff-Love and Reissner-Mindlin shell formulations, C^1-continuity is required, which can also be satisfied by alternative discretization, like subdivision surfaces. Additionally, model adaptivity is facilitated since a switch between a Kirchhoff-Love and Reissner-Mindlin formulation is possible without any changes in the parameterization. Future research deals with a more thorough investigation of patch coupling schemes, extension to trimmed NURBS or further advanced discretization schemes.

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