A Discretization Independent Methodology for Mixed Methods

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Micro Abstract

Nowadays a broad range of different discretization methods, such as (isogeometric) spline based finite element approaches, collocation or meshless methods are available to compute approximate solutions for boundary value problems. Locking is a common issue for primal formulations in all these schemes and formulations based on mixed methods may be favorable. A general methodology will be presented to construct necessary strain/stress ansatz spaces, independent of the discretization method.

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Introduction

Numerical methods to solve problems in the field of structural mechanics are often characterized by locking effects and special formulations have to be developed to overcome this issue. There exist a vast amount of publications on this topic dealing with formulations to avoid locking. However, the majority of these ideas depend on the discretization and are only valid for very specialized formulations, e.g. four-node 2D elements in the context of the finite element method, dealing with one specific locking phenomenon, e.g. shear locking.

Mixed methods are a common approach to overcome the issue of locking but they usually require proper choices of function spaces, e.g. for stress or strain variables, to remove locking and avoid artificial oscillations. Thus for each discretization scheme proper (lower) function spaces have to be constructed. The present concept, however, does not require specific construction of feasible spaces. Instead, all variables are discretized by equal-order interpolation and the same function space may be used for all variables. Due to the specific construction of the principle, the result is automatically locking-free and does not exhibit any spurious oscillations. The present contribution is based on the work of [2].

1 Variational Framework

In order to explain the basic idea of the proposed method we introduce the modified Hellinger-Reissner functional

$$\Pi_{\rm HR}\left(\mathbf{u}, \mathbf{E}\right) = \int_{\Omega} \left(\frac{1}{2}\mathbf{E}^{\rm T}\mathbf{C}\mathbf{E} - \mathbf{E}_{\rm u}^{\rm T}\mathbf{C}\mathbf{E}\right) \,\mathrm{d}\Omega + \Pi_{\rm HR}^{\rm ext}(\mathbf{u}) \tag{1}$$

with displacements \mathbf{u} and Green-Lagrange strains \mathbf{E} as two unknown fields. \mathbf{C} denotes the linear elasticity material tensor and \mathbf{E}_u the Green-Lagrange strain tensor derived from the displacements \mathbf{u} via a differential operator

$$\mathbf{E}_{\mathrm{u}} = \mathcal{L}\mathbf{u}.\tag{2}$$

The main idea of the proposed method is to introduce displacement–like variables $\bar{\mathbf{u}}$ and a new differential operator $\bar{\mathcal{L}}$, defined below. The independent strain field \mathbf{E} can now be derived via

$$\mathbf{E} =: \mathbf{E}_{\bar{\mathbf{u}}} = \bar{\mathcal{L}} \bar{\mathbf{u}} \tag{3}$$

and we can rewrite the internal parts of equation (1) as

$$\Pi_{\mathrm{HR}}^{\mathrm{int}}\left(\mathbf{u},\bar{\mathbf{u}}\right) = \int_{\Omega} \left(\frac{1}{2}\mathbf{E}_{\bar{\mathrm{u}}}^{\mathrm{T}}\mathbf{C}\mathbf{E}_{\bar{\mathrm{u}}} - \mathbf{E}_{\mathrm{u}}^{\mathrm{T}}\mathbf{C}\mathbf{E}_{\bar{\mathrm{u}}}\right) \,\mathrm{d}\Omega \tag{4}$$

with \mathbf{u} and $\bar{\mathbf{u}}$ as independent variables. The variation of equation (4) yields

$$\delta\Pi_{\mathrm{HR}}^{\mathrm{int}}\left(\mathbf{u},\bar{\mathbf{u}}\right) = \int_{\Omega} \left(\delta \mathbf{E}_{\bar{\mathbf{u}}}^{\mathrm{T}} \mathbf{C} \mathbf{E}_{\bar{\mathbf{u}}} - \delta \mathbf{E}_{\mathbf{u}}^{\mathrm{T}} \mathbf{C} \mathbf{E}_{\bar{\mathbf{u}}} - \mathbf{E}_{\mathbf{u}}^{\mathrm{T}} \mathbf{C} \delta \mathbf{E}_{\bar{\mathbf{u}}} \right) \, \mathrm{d}\Omega.$$
(5)

Since two displacement(-like) variables appear in the functional (5), we denote this concept as "mixed displacement" (MD) method. The essential advantage of the method proposed herein reveals for a discretization of equation (5). The typical challenge in the context of mixed finite elements of identifying proper ansatz spaces for the newly introduced fields, e.g. strain or stress fields, is obsolete. A discretization on the basis of equation (5) is locking-free a priori and equal order interpolation can be used for all displacement fields \mathbf{u} and $\bar{\mathbf{u}}$

$$\mathbf{u}^{\mathrm{h}} = \sum_{i=1}^{n} N^{i} \mathbf{d}^{i}, \qquad \qquad \bar{\mathbf{u}}^{\mathrm{h}} = \sum_{i=1}^{n} N^{i} \bar{\mathbf{d}}^{i}. \tag{6}$$

Here, \mathbf{d}^i and $\bar{\mathbf{d}}^i$ denote the nodal degrees of freedom and N^i are the shape functions. Equation (6) with (3) effectively establishes a method where the physical entities of the discretized variables do not coincide with the entities of the additionally introduced field. For this particular case we have $\mathbf{E}^{\rm h} = \bar{\mathcal{L}} \mathbf{N} \bar{\mathbf{d}}$, i.e. an independent *strain* field is interpolated from nodal quantities with the physical unit of *displacements*.

Remaining issues are the selection of the new variables $\bar{\mathbf{u}}$ and the definition of the corresponding differential operator $\bar{\mathcal{L}}$. This can be achieved considering the DSG method introduced in [3], from which the idea of this formulation was inspired. The additional displacement-like field $\bar{\mathbf{u}}$ is chosen as primitive of those strains that are the source of locking. Thus, the field $\bar{\mathbf{u}}$ can be interpreted as the shear/strain gaps from the DSG method. Note that the introduced displacement field $\bar{\mathbf{u}}$ is not unique and proper constraint conditions have to be applied, in order to remove the modes for which $\bar{\mathcal{L}}\bar{\mathbf{u}} = \mathbf{0}$. The procedure is practically identical to the one described earlier in [5].

2 Model Problem: Timoshenko beam

The variational method proposed herein is first be explained for a simple 1-D Timoshenko beam. The primary variables and the kinematic equations to derive the curvature and shear angle for a standard Timoshenko beam formulation with vertical displacement v and the total rotation of the cross section φ read

$$\mathbf{u} = \begin{bmatrix} v \\ \varphi \end{bmatrix} \quad \text{and} \quad \mathbf{E}_{\mathbf{u}} = \begin{bmatrix} \gamma \\ \kappa \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 1 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} v \\ \varphi \end{bmatrix} = \begin{bmatrix} v_{,x} + \varphi \\ \varphi_{,x} \end{bmatrix}.$$
(7)

Here, the unbalance in the kinematic equation to derive the shear angle is responsible for the well known transverse shear locking effect. Following the idea of section 1 we introduce an additional displacement field $\bar{\mathbf{u}}$ and define the new kinematic equation as

$$\bar{\mathbf{u}} = \begin{bmatrix} \bar{v}_{\mathrm{s}} \\ \varphi \end{bmatrix} \quad \text{and} \quad \mathbf{E}_{\bar{\mathrm{u}}} = \begin{bmatrix} \bar{\gamma} \\ \kappa \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix}}_{\bar{\mathcal{L}}} \begin{bmatrix} \bar{v}_{\mathrm{s}} \\ \varphi \end{bmatrix} = \begin{bmatrix} \bar{v}_{\mathrm{s},x} \\ \varphi_{,x} \end{bmatrix}$$
(8)

Note that κ is not treated in a special way, since there is no unbalance present that would cause any locking. Inserting equation (7) and (8) into (5) yields

$$\delta\Pi_{\mathrm{HR}}^{\mathrm{int}}\left(\mathbf{u},\bar{\mathbf{u}}\right) = \int_{\Omega} \left[\delta\bar{v}_{\mathrm{s},x}\mathrm{GA}\left(\bar{v}_{\mathrm{s},x} - v_{,x} - \varphi\right) - \left(\delta v_{,x} + \delta\varphi\right)\mathrm{GA}\bar{v}_{\mathrm{s},x} - \delta\varphi_{,x}\mathrm{EI}\varphi_{,x}\right]\,\mathrm{d}\Omega\tag{9}$$



Figure 1. Uni-axial plate bending, problem setup (NURBS mesh with 6 elements and 8 control points).

Using the Euler-Lagrange equations of the weak form (9) we can eliminate $\varphi_{,x}$ and exactly reproduce the hierarchic displacement formulation from Oesterle et al. [6], where $v_{\rm s}$ (here denoted as $\bar{v}_{\rm s}$) is introduced as extra primary variable in addition to v.

In general, the construction of the differential operator $\overline{\mathcal{L}}$ follows this rule: Each strain component has its own "strain displacement" and the strains are obtained by a first derivative of the corresponding displacement variable.

3 Numerical Examples

A simple uni-axial plate problem as shown in Figure 1 is presented in order to show that the proposed variational formulation effectively removes shear locking-phenomena, stress resultants are free from oscillations and the method is independent of the discretization. To do so, three different discretization schemes are utilized:

- Quadrilateral standard finite elements with bilinear shape functions (Q1).
- Isogeometric, quadratic NURBS-based finite elements as introduced in [4].
- A meshless discretization scheme based on maximum entropy (max-ent) approximants introduced in [1].

With these schemes, three different formulations are discretized.

- Mindlin–u: Standard displacement based (primal) Mindlin plate formulation, utilizing one mid-surface displacement v and two rotations φ_x and φ_y .
- Mindlin–MD: Mindlin plate formulation based on the mixed displacement (MD) concept, as described in Section 2.
- Mindlin–u-E: Mindlin plate formulation based on the two-field formulation with displacement and strain ansatz according to equation (1). Note that for the sake of comparability we utilize equal order interpolation for all displacement and strain parameters, although lower order spaces for the strains may be found for some of the applications shown here that would exhibit better properties.

Figures 2 shows numerical results obtained with the different discretization schemes mentioned before. All discretizations use the same number of nodes or control points in x-direction. On the left side of Figure 2 typical locking diagrams are shown in which the maximum displacement is plotted versus a critical parameter, here the slenderness L_x/t . The results are compared to the exact thin beam solution, labelled as "Bernoulli". On the right hand side of the mentioned



Figure 2. Uni-axial plate bending, different discretization schemes, 8 nodes/control points in x-direction; left: maximum displacement v_{max} vs. slenderness; right: transverse shear force q_1 along x-direction for slenderness $\frac{L_x}{t} = 100$.

figures the shear force distribution is shown for a specific, moderate slenderness of $L_x/t = 100$ in order to display the influence of locking on the quality of the corresponding stress resultant.

The standard formulation Mindlin–u suffers severely from transverse shear locking, which reveals in a significant underestimation of the displacements in the thin regime. The effect is most pronounced for linear standard finite elements (not shown), but it is present also for isogeometric analysis with quadratic NURBS and the meshless approach using max-ent exponential functions. The shear forces obtained with the standard displacement formulation Mindlin–u oscillate in an unacceptable manner for all discretization schemes. This highlights the importance of investigating stress quality and not only displacements when assessing the performance of discretization schemes with regard to locking.

Both mixed formulations Mindlin–u-E and Mindlin–MD show accurate results for displacements, independent of the slenderness and the discretization scheme. Concerning the shear forces, Mindlin–MD shows superior quality for the isogeometric and the meshless discretization. The reason is that for equal order interpolation the strain spaces of the Mindlin–u-E method are too rich and thus oscillations in the stress resultants still show up.

For further numerical examples including the issue of membrane locking in geometrical linear and non-linear shell problems we refer to [2].

Conclusions

In this contribution a unified concept to tackle all geometrical locking effects in solid and structural mechanics, independent of the discretization scheme and the underlying structural model, has been presented. The method is denoted as mixed displacement (MD) method, as it involves displacement degrees of freedom \mathbf{u} as well as additional "displacement-like" degrees of freedom $\mathbf{\bar{u}}$. The latter are used to discretize strains (stresses, respectively) in a two-field Hellinger-Reissner principle.

Due to its generality, the MD method has the potential to be applied in cases with highly unstructured function spaces, e.g. when using T-splines within isogeometric analysis. In such cases, appropriate function spaces for standard mixed methods or assumed strain methods may be hard to find. The fundamental motif of using discrete values with a different physical unit than the actual field that they represent may be interesting also in other fields outside solid and structural mechanics.

Current investigations focus on the observed relationship of the method to hierarchic displacement formulations, the appropriate treatment of the additional constraint conditions for the $\bar{\mathbf{u}}$ -field as well as a thorough study of the mathematical background of the method.

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