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A shell element for the analysis of interlaminar stresses and delaminations of layered composites

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Micro Abstract
In this contribution a mixed hybrid shell element for the calculation of interlaminar shear and normal thickness stresses of layered composite structures is presented. These stresses are decisive factors for delaminations. The element formulation is based on the Reisser-Mindlin kinematics with an inextensible director field. After static condensation the element has the standard shell degrees of freedom. The numerical examples focus on failure caused by delamination.

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Introduction

Composite materials become more important in a wide range of industrial manufacturing. Therefore effective numerical models are essential for the simulation of such materials. Especially failure modes of laminates are of great interest. Here, interlaminar stresses, such as shear and normal thickness stresses, are decisive factors for delamination. There are different possibilities to compute these stresses. The use of solid or solid-shell elements [3], which model a full 3D stress state are an obvious approach. In order to obtain interlaminar stresses with solid elements, a minimum of 3 to 5 elements per layer are required. This leads to high numerical costs for the computation of large multi-layered structures. Higher order shell elements are another possibility, but they lead to problems when defining the boundary conditions of complex geometries. The proposed shell formulation overcomes these problems. Based on [2], the element formulation is expanded to geometrical non-linearity and the calculation of normal stresses in thickness direction, see [1].

1 Shell kinematics and interpolation in thickness direction

The underlying shell kinematics are based on the Reissner-Mindlin theory with an inextensible director field. Derived from the Green-Lagrangian strain tensor, the membrane strains ε, curvatures κ and transverse shear strains γ are introduced and summarized in the vector

$$\boldsymbol{\varepsilon}_g(\mathbf{v}) = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}, \kappa_{11}, \kappa_{22}, 2\kappa_{12}, \gamma_1, \gamma_2]^T \,. \tag{1}$$

The superposed displacement field $\tilde{\mathbf{u}} = \tilde{u}_i \mathbf{t}_i$ is introduced, with components relating to a local base system $\mathbf{t}_i$. The vector $\tilde{u}_\alpha$ with $\alpha = 1, 2$ describes the out of plane warping displacements and $\tilde{u}_3$ the relative thickness displacements. The shape of $\tilde{\mathbf{u}}$ through the thickness is chosen as

$$\tilde{\mathbf{u}}(\xi^3) = \boldsymbol{\Phi}(\boldsymbol{\xi^3})\,\boldsymbol{\alpha}\,. \tag{2}$$

The vector $\boldsymbol{\alpha}$ contains displacements at the nodes through the thickness of the laminate, see Fig. 1. The number of the components of $\boldsymbol{\alpha}$ depends on the number of layers N. The warping and

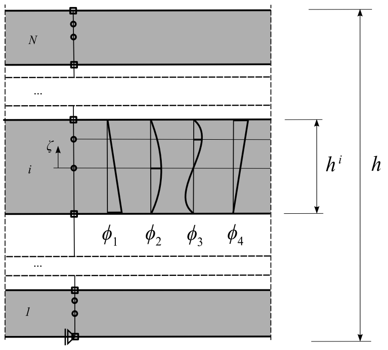

Figure 1. Interpolation functions for laminate with N layers

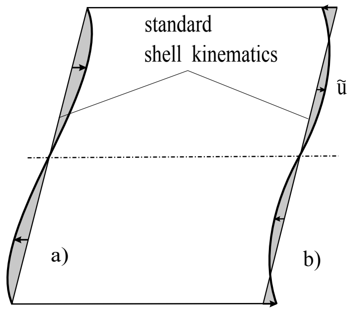

Figure 2. a) wrong and b) correct shape of warping displacements

relative thickness displacements are elementwise constant and interpolated by cubic hierarchic functions

$$\begin{aligned} &\mathbf{\Phi}(\xi^{\mathbf{3}}) = \left[\ \phi_1\,\mathbf{1_3} \quad \phi_2\,\mathbf{1_3} \quad \phi_3\,\mathbf{1_3} \quad \phi_4\,\mathbf{1_3}\ \right]\,\mathbf{a^i} \\ &\phi_1 = \frac{1}{2}\,(1-\zeta) \qquad \phi_2 = 1-\zeta^2 \qquad \phi_3 = \frac{8}{3}\,\zeta\,(1-\zeta^2) \qquad \phi_4 = \frac{1}{2}\,(1+\zeta)\,, \end{aligned} \tag{3}$$

where $-1 \leq \zeta \leq 1$ is a normalized thickness coordinate of layer i. Furthermore, $\mathbf{a^i}$ is an assembly matrix, which relates the 12 degrees of freedom of layer i to the components of $\boldsymbol{\alpha}$, while $\mathbf{1_n}$ refers to the unity matrix. The total displacements of the shell

$$\hat{\mathbf{u}} = \mathbf{u} + \xi^{\mathbf{3}}\,(\mathbf{d} - \bar{\mathbf{D}}) + \tilde{\mathbf{u}} \tag{4}$$

are obtained by superposition of the averaged displacements of the Reissner–Mindlin theory with the fluctuation field (2). In (4) $\bar{\mathbf{D}}$ and $\mathbf{d}$ denote to the normal vector in reference and current configuration respectively. The layer strains of a point in shell space with coordinates ξ^3 are obtained from

$$\mathbf{E} = \mathbf{A}_1\,\boldsymbol{\varepsilon} + \mathbf{A}_2\,\boldsymbol{\alpha} = [E_{11}, E_{22}, E_{33}, 2E_{12}, 2E_{13}, 2E_{23}]^T \tag{5}$$

where the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ contain the thickness coordinate and the differentiation of the cubic interpolation functions respectively. The first part $\mathbf{A}_1\,\boldsymbol{\varepsilon}$ refers to the physical shell strains and the second part $\mathbf{A}_2\,\boldsymbol{\alpha}$ to the superposed displacement field $\tilde{\mathbf{u}}$.

2 Weak form of boundary value problem

The equilibrium of stresses and higher order stress resultants leads, with admissible variations $\delta\boldsymbol{\theta} := [\delta\mathbf{v}, \delta\boldsymbol{\sigma}, \delta\boldsymbol{\varepsilon}, \delta\boldsymbol{\alpha}, \delta\boldsymbol{\lambda}]^T$ with $\delta\mathbf{v} := [\delta\mathbf{u}, \delta\boldsymbol{\varphi}]^T$ to the weak form of the boundary value problem

$$\begin{aligned} g(\boldsymbol{\theta}, \delta\boldsymbol{\theta}) &= \int_\Omega [\delta\boldsymbol{\varepsilon}_g^T\,\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}_g - \boldsymbol{\varepsilon}) + \delta\boldsymbol{\varepsilon}^T\left(\partial_{\boldsymbol{\varepsilon}} W - \boldsymbol{\sigma}\right) \\ &\quad + \delta\boldsymbol{\alpha}^T\left(\partial_{\boldsymbol{\alpha}} W + \mathbf{D}_{23}\,\boldsymbol{\lambda} - \bar{\mathbf{q}}\right) + \delta\boldsymbol{\lambda}^T\mathbf{g}]\,\mathrm{d}A + g_{ext} = 0 \\ g_{ext} &= -\int_\Omega \delta\mathbf{u}^T\bar{\mathbf{p}}\,\mathrm{d}A - \int_{\Gamma_\sigma} \delta\mathbf{u}^T\,\bar{\mathbf{t}}\,\mathrm{d}s\,, \end{aligned} \tag{6}$$

where $\boldsymbol{\lambda}$ containing the derivatives of strains and curvatures, In (6) $\boldsymbol{\sigma}$ refers to the vector of stress resultants, $\partial_{\boldsymbol{\varepsilon}} W$ and $\partial_{\boldsymbol{\alpha}} W$ to the differentiation of the strain energy density with respect to shell strains and the additional through thickness displacements $\boldsymbol{\alpha}$ respectively. The weak form contains the constraint $\mathbf{g}(\boldsymbol{\alpha}) = \mathbf{D}_{23}\boldsymbol{\alpha}$, which enforces the correct shape of warping, Fig. 2.

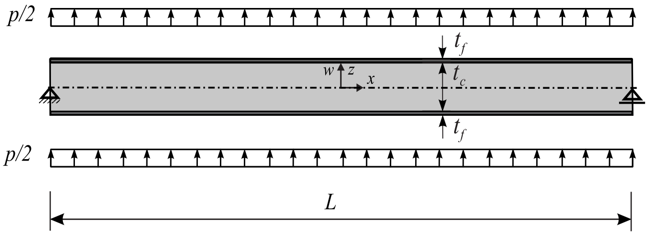

Figure 3. Sandwich beam with constant loading

Material	Geometry
$E_c = 70\,\mathrm{N/mm^2}$	$p = 1.0\,\mathrm{N/mm^2}$
$\nu c = 0.3$	$L = 2000\,\mathrm{mm}$
$E_f = 70000\,\mathrm{N/mm^2}$	$t_c = 30\,\mathrm{mm}$
$\nu_f = 0.3\,\mathrm{N/mm^2}$	$t_f = 0.5\,\mathrm{mm}$
$y_0 = 100\,\mathrm{N/mm^2}$	$b = 60\,\mathrm{mm}$
$\xi = 1000\,\mathrm{N/mm^2}$	$b = 60\,\mathrm{mm}$

Figure 4. Material and geometrical properties

Because of the later use of the Newton interation, the weak form in Eq. 6 must be linearized. This leads to a set of equations

$$\mathrm{L}\left[g(\boldsymbol{\theta},\delta\boldsymbol{\theta}),\Delta\boldsymbol{\theta}\right] := g(\boldsymbol{\theta},\delta\boldsymbol{\theta}) + \mathrm{D}g\cdot\Delta\boldsymbol{\theta} = g_{ext} + \int_\Omega \Delta\delta\varepsilon_g^T\sigma\,\mathrm{d}A$$
$$+\int_\Omega \begin{bmatrix}\delta\boldsymbol{\varepsilon}_g\\ \delta\boldsymbol{\sigma}\\ \delta\boldsymbol{\varepsilon}\\ \delta\boldsymbol{\alpha}\\ \delta\boldsymbol{\lambda}\end{bmatrix}^T \left\{\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\varepsilon}_g-\boldsymbol{\varepsilon}\\ \partial_{\boldsymbol{\varepsilon}}W-\boldsymbol{\sigma}\\ \partial_{\boldsymbol{\alpha}}W+\mathbf{D}_{23}\,\boldsymbol{\lambda}-\bar{\mathbf{q}}\\ \mathbf{g}\end{bmatrix} + \begin{bmatrix}\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&-\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&-\mathbf{1}&\mathbf{D}_{11}&\mathbf{D}_{12}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{D}_{21}&\mathbf{D}_{22}&\mathbf{D}_{23}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{D}_{32}&\mathbf{0}\end{bmatrix}\begin{bmatrix}\Delta\boldsymbol{\varepsilon}_g\\ \Delta\boldsymbol{\sigma}\\ \Delta\boldsymbol{\varepsilon}\\ \Delta\boldsymbol{\alpha}\\ \Delta\boldsymbol{\lambda}\end{bmatrix}\right\}\mathrm{d}A \tag{7}$$

where

$$\begin{bmatrix}\Delta\,\partial_{\boldsymbol{\varepsilon}}W\\ \Delta\,\partial_{\boldsymbol{\alpha}}W\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{11}&\mathbf{D}_{12}\\ \mathbf{D}_{21}&\mathbf{D}_{22}\end{bmatrix}\begin{bmatrix}\Delta\,\boldsymbol{\varepsilon}\\ \Delta\,\boldsymbol{\alpha}\end{bmatrix} \qquad \mathbf{D}_{\alpha\beta} = \int_{h^-}^{h^+}\mathbf{A}_\alpha^T\,\mathbf{C}\,\mathbf{A}_\beta\,\bar{\mu}\;\mathrm{d}\xi^3 \tag{8}$$

which is the basis for the further FEM formulation. Matrix $\mathbf{C}$ refers to the material matrix for the assumption of orthotropic material behavior.

3 Finite element formulation

The finite element formulation is based on the isoparametric concept for quadrilateral shell elements with bilinear form functions. Interpolation functions for displacements and independent quantities are chosen. Latter are eliminated by static condensation, which leads to the standard shell degrees of freedom. Nodes at intersection have 6 DOFs (3 displacements, 3 rotations), all other nodes have 5 DOFs (3 displacements, 2 rotations).

4 Examples

4.1 Sandwich strip

In the first example the results of a sandwich strip with constant loading on the upper and lower surface are shown, see Fig. 3 The material properties are summarized in Fig. 4. A geometrically and physically non-linear computation is carried out and the results for normal thickness stresses Fig. 5 and transverse shear stresses Fig. 6 are shown for load factor $\lambda = 4$, where plastification occurred. Further Fig. 7 shows the load-displacement curve at $x = 0$ for the loading and unloading of the sandwich strip. The results show good agreement with the reference solution, computed with 3d solid shell elements.

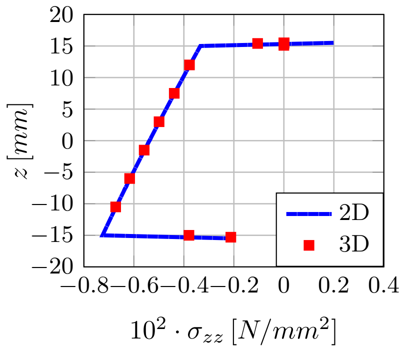

Figure 5. σ_{zz} at $x = 390$ mm for $\lambda = 4$

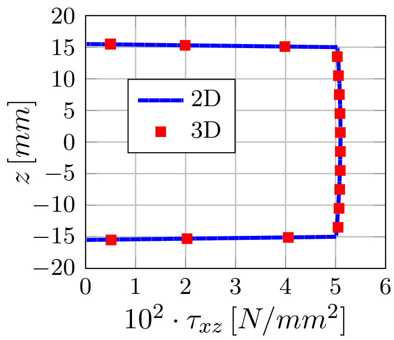

Figure 6. τ_{xz} at $x = 390$ mm for $\lambda = 4$

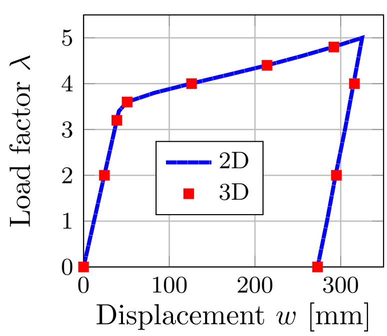

Figure 7. Load factor vs. displacement at $x = 0$

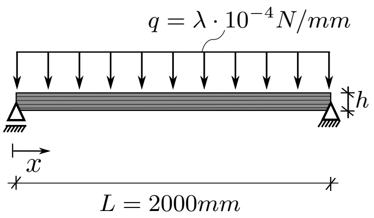

Figure 8. Beam with constant loading and $h = b = 10mm$

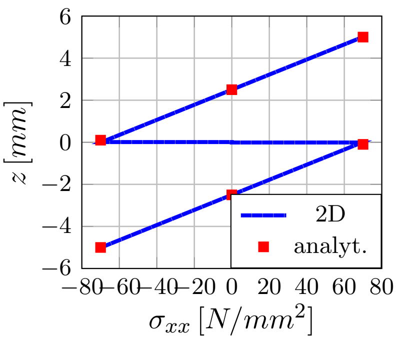

Figure 9. σ_{xx} at $x = 1500$ mm for $\lambda = 15$

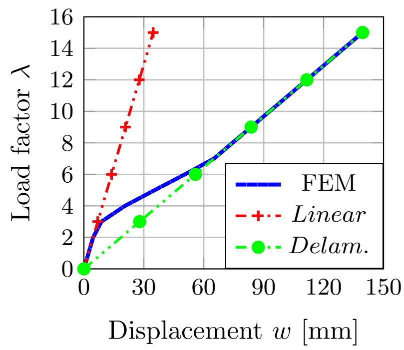

Figure 10. Load factor vs. displacement at $x = L/2$

4.2 Delamination

This example shows a beam with constant load with 10 isotropic layers, where $E = 10^5 N/mm^2$, $\nu = 0.4$, fracture energy $G_f = 10^{-4} N/mm$ and the ultimate stress $Y_0 = 10^{-5} N/mm^2$, see Fig. 8. While the load factor λ is increased, delamination due to shear stresses occurs in the middle layer and spreads along the beam. This leads to a decrease of stiffness and a jump of interlaminar normal stresses, see Fig. 9. The load-displacement curve is shown in Fig. 10, as well as the reference functions for the beam without (*Linear*) and with full delamination (*Delam.*).

Conclusions

In this abstract a finite element shell formulation for the computation of layered structures is shown. Interlaminar shear and normal stresses are obtained for linear, as well as geometrically and physically non-linear problems. The presented shell element shows good agreement with the solution of 3d models and overcomes the problems of these numerical expensive computations.

References

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