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Smooth spline spaces on unstructured quadrilateral meshes for isogeometric analysis

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Micro Abstract
We present a framework for isogeometric analysis on unstructured quadrilateral meshes. Acknowledging the differing requirements posed by design and analysis, we propose the construction of a separate, smooth spline space for each, while ensuring isogeometric compatibility. A key ingredient in the approach is the use of singular parameterizations at extraordinary vertices. We demonstrate the versatility of the approach with applications in design and analysis.

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Introduction

Modeling arbitrary genus geometries with a finite number of tensor-product patches invariably leads to surface representations over unstructured quadrilateral meshes containing *extraordinary points*, i.e., internal vertices where $\mu \neq 4$ edges meet. On regular parts of the mesh, where the quadrilateral elements are arranged in a locally-structured fashion, smooth splines can be easily built. However, there is no canonical way of doing the same on an unstructured arrangement of quadrilateral elements. Application of smooth splines over unstructured meshes is of considerable interest within the field of isogeometric analysis [1], and a myriad of approaches have been explored that focus on the design and analysis of geometries built over such meshes; see, e.g., [4].

Taking inspiration from [2,3], we present a novel framework for construction of smooth splines on unstructured quadrilateral meshes, providing a solution in the context of both geometric modeling and isogeometric analysis. Geometric modeling is a highly visual process and is driven by practical considerations. It is important to have simple, intuitive modeling tools that can be used to build visually pleasing geometries. Computationally analyzing the designed geometric objects, on the other hand, is very much motivated by accuracy and stability considerations. Keeping the above in mind, we acknowledge the differing requirements posed by design and analysis and propose a two-pronged solution. We aim to construct two spline spaces, called the *design* and *analysis spaces*, for the purposes of geometric modeling and isogeometric analysis, respectively. The general requirements imposed on the functions that span each space are summarized in Figure 1. We refer the reader to [6] for full details on the complete framework.

1 Construction of Unstructured Spline Spaces

We will build spline spaces suitable for design and analysis over a mesh $\mathcal{M}$ that is *standardized*, i.e., for any given extraordinary point, its spoke edges have the same knot spans. We focus on the construction of smooth, linearly independent, locally supported spline functions over $\mathcal{M}$. The design and analysis spline spaces are then defined as the spans of these splines. For the sake of simplified exposition, we restrict ourselves to bi-cubic splines, but the ideas can be extended to higher degrees. The construction of smooth spline basis functions over $\mathcal{M}$ is essentially done in a two-step process [6]:

Spline basis

Non-negativity[1]

Partition of unity

Linear independence

Local support

Geometric representation		**Analysis-suitable representation**
At least C^1 smoothness		At least C^1 smoothness
Finite representation	$\mathbb{S}_D \subseteq \mathbb{S}_A$	Finite representation
Intuitive control net		Optimal approximation
Small consistency error		Nestedness[1]

Compatibility with CAD technology (T-splines,...)

Figure 1. A summary of the properties achievable within our unstructured spline framework. For the specific task of building geometric models, the spline basis (spanning the design space, $\mathbb{S}_D$) is, in particular, associated with an intuitive control net and maximizes the regions of C^2 smoothness; this necessarily entails non-nested geometries under refinement during geometric modeling. Once the geometric model has been built, an analysis space, $\mathbb{S}_A$, is built on the same mesh ensuring isogeometric compatibility, $\mathbb{S}_D \subseteq \mathbb{S}_A$. Thereafter, for performing analysis on the geometry, it is possible to build a nested sequence of spaces such that the geometric model stays invariant.

1. **Macro extraction**: For each element $\omega \in \mathcal{M}$, we construct a linear map from spline degrees of freedom to ω-local Bézier degrees of freedom. Such a linear map is called the *Bézier extraction operator*, and its transpose the *macro spline extraction operator*.

2. **Micro extraction**: Spline basis functions defined using the macro extractions on each element will be C^2 smooth except on the 1-ring elements around the extraordinary point, called *irregular elements*. We will rectify this using the *split-then-smoothen* approach:

   (a) split each irregular element into 2×2 sub-elements using de Casteljau's algorithm;

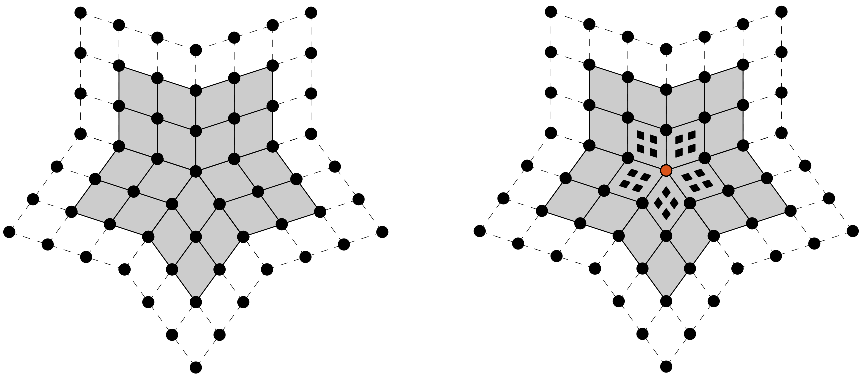
   (b) transform the inner Bézier sub-elements adjacent to the extraordinary point into singular but smooth *D-patches*, using a so-called *smoothing matrix*.

   This will yield spline basis functions that are linear combinations of Bernstein basis functions defined on the 2×2 sub-elements of the split 1-ring elements, and the corresponding linear map is called the *micro spline extraction operator*.

A single spline function is thus built for each spline degree of freedom. For geometric modeling, conforming with the norm, the degrees of freedom are vertex-based control points. For isogeometric analysis, some of the vertex-based control points are excluded and new ones on element interiors are introduced. Note that macro extractions depend on the particular choice of spline degrees of freedom; details can be found in [6]. The corresponding control structures are illustrated in Figures 2a and 2b, respectively, and the associated spline functions lead to the so-called *design* and *analysis spaces*, respectively. Moreover, the construction of $\mathbb{S}_D$ and $\mathbb{S}_A$ are such that *isogeometric compatibility*, $\mathbb{S}_D \subseteq \mathbb{S}_A$, is ensured.

The framework is enacted as follows: geometric models are built using $\mathbb{S}_D$ and, once finalized, are represented as members of $\mathbb{S}_A$; this representation is exact since $\mathbb{S}_D \subseteq \mathbb{S}_A$. Subsequently,

[1]Contingent upon the properties of the smoothing matrix [6].



(a) Design degrees of freedom **(b)** Analysis degrees of freedom

Figure 2. These figures illustrate spline control nets (near extraordinary points) used for geometric modeling and isogeometric analysis. Active control points have been plotted in black and inactive ones in red. For design only vertex-based control points are used, while for analysis we introduce face-based control points near extraordinary points and forgo reliance on traditional control nets in favor of creating nested spaces.

since nested spaces can be built in the analysis phase, the geometry can be exactly preserved when refining during the analysis phase. Other features of the design and analysis spaces are shown in Figure 1.

2 Application in Design and Analysis

We now consider a Cahn–Hilliard problem over the surface of a double-doughnut Ω. The problem models spinodal decomposition of a binary fluid and is governed by a fourth-order non-linear PDE. The surface, modeled using the design space, has been illustrated in Figure 3. For solving the PDE, we switched to the analysis space and performed a few steps of global refinement in order to satisfactorily resolve the evolving interface. The following non-dimensional form of the problem was solved:

$$\frac{\partial c}{\partial t} = \nabla_\Omega \cdot (c(1-c)\nabla_\Omega(\mathbb{N}_2\mu_c - \Delta_\Omega c)) \quad \text{on } \Omega \times [0,\ T]\ ,$$
$$c(\boldsymbol{x}, 0) = c_0(\boldsymbol{x}) \quad \text{on } \Omega\ ,$$

where ∇_Ω and Δ_Ω are the surface gradient and Laplace–Beltrami operators, respectively, and $\mu_c := \frac{1}{3}\log\left(\frac{c}{1-c}\right) + 1 - 2c$. The analysis mesh had $12,382$ degrees of freedom, and the initial value c_0 was determined by randomly perturbing the chosen initial volume fraction $\bar{c} = 0.5$; the corresponding value of $\mathbb{N}_2$ was $3,282.5$. The results are shown in Figure 4. Steady state was reached for the configuration in 650 time-steps.

Further examples in design and analysis are described in [6].

Conclusions

We presented a C^1 construction of bi-cubic splines on unstructured quadrilateral meshes containing extraordinary points. Appreciating the differing requirements posed by geometric modeling and isogeometric analysis, separate spline spaces for both fields were built, and their suitability for applications in these fields was demonstrated.

A key feature of our construction is its locality, making it highly portable. For example, it would be straightforward to combine the construction here with the one in [5] for smooth polar splines. Additionally, performing such a construction in the context of locally refinable spline technologies (based on local tensor-product structures) should be straightforward; see [6] for its integration in the T-spline framework.

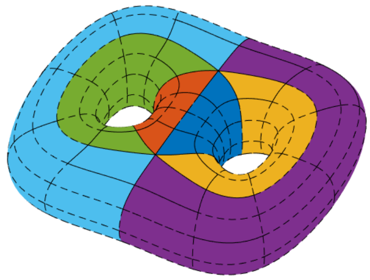
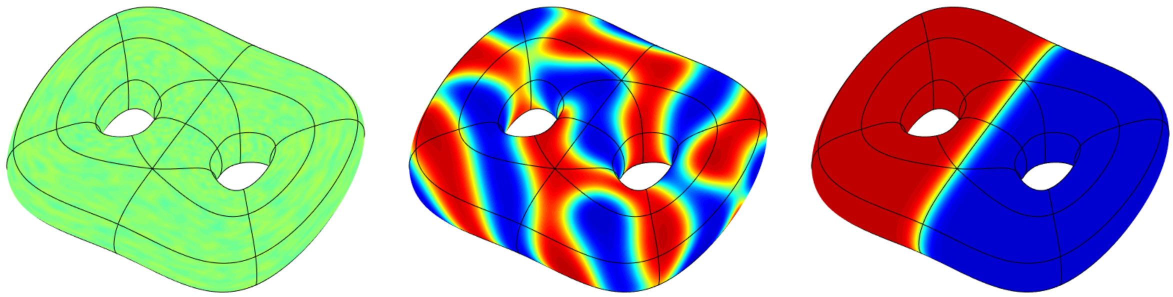


Figure 3. The surface of a double-doughnut, modeled in the design space, is the domain of interest for the Cahn–Hilliard problem. The iso-parameter lines and colors indicate the locations and valences (6) of the extraordinary points.



(a) $t = 0$ **(b)** $t = 4.3197 \times 10^{-5}$ **(c)** $t = 1006.4156$

Figure 4. The initial volume-fraction distribution is shown together with its time-evolution for the Cahn–Hilliard problem over the surface in Figure 3. The meshes used for the computation contained $12,382$ degrees of freedom.

Acknowledgements

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