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Optimal control of a slot car racer

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Micro Abstract
Within this work, we compute and apply control strategies for the time-minimal path of a slot car racer. Here, the DMOC (Discrete Mechanics and Optimal Control) method is used to generate offline optimal trajectories for the electro-mechanically coupled system, i.e. sequences of discrete configurations and driving voltages. These sequences are embedded to a control architecture with an underling camera tracking system which allows to correct the vehicle towards the desired state via computer.

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Introduction

In order to describe and control the behavior of a slot car racer, a suitable simulation model is required. The functional principle of this electric vehicle merges mechanics and electronics and can generally be described in terms of differential equations by physical laws, such as Faraday's law, Coulomb's law, Kirchhoff's law and d'Alembert's principle. These second-order differential equations can be obtained via a variational principle based on an energy functional [1, 6].
The optimal control simulation method in this work is a direct discretization technique for mechanical systems – that has been extended for mechatronical systems – known as DMOC [4] and is based on a discrete variational principle. The derivation of the system dynamics with discrete variational calculus requires to formulate the electrical, magnetic and mechanical energy of the system and to apply the discrete Lagrange-d'Alembert principle. This is less common in electrical engineering but leads to a structure preserving time stepping scheme which serves as equality constraints for the nonlinear programming problem, resulting from the discretization of the optimal control problem by DMOC [2–5].
The computed optimal voltage profiles are embedded into an experimental setup for a slot car racer with an underlying camera tracking system which allows to correct the vehicle towards the desired state via a computer. Furthermore, the tracking allows to analyze the system, fit the model parameters and measure the maximal admissible velocity for the race track, which also serve as constraints for the optimal control problem.

1 Discrete mechanics and optimal control

In this section, we present a simulation method for the optimal control of a mechatronic system that is based on a discrete variational principle and apply it to compute the time-minimal path of a slot car racer. Here, DMOC makes use of the discrete forced Euler-Lagrange equations to generate offline optimal trajectories for the electro-mechanically coupled system.

1.1 Discrete forced Euler-Lagrange equations

In general, the discrete variational principle yields discrete time stepping equations. Their solution approximates the solution of the forced Euler-Lagrange equations and inherits certain characteristic properties of the continuous solution. According to the discrete variational principle, we choose a time grid $\Delta t = \{t_0, t_1, ..., t_N\}$ for the discrete path $\mathbf{q}_d = \{\mathbf{q}_n\}_{n=0}^{N}$ with step size $h \in \mathbb{R}$ and the midpoint rule for the approximation of the integrals in the Lagrange-d'Alembert principle.

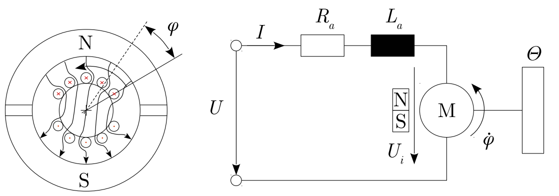

Figure 1. Idealized model of a DC motor

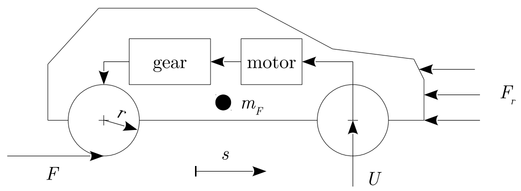

Figure 2. Slot car model

The discrete forced Euler-Lagrange equations for $n = 1, ..., N-1$

$$D_1 L_d(\mathbf{q}_n, \mathbf{q}_{n+1}) + D_2 L_d(\mathbf{q}_{n-1}, \mathbf{q}_n) + \mathbf{f}_d^-(\mathbf{q}_n, \mathbf{q}_{n+1}, \mathbf{u}_n) + \mathbf{f}_d^+(\mathbf{q}_{n-1}, \mathbf{q}_n, \mathbf{u}_n) = \mathbf{0} \tag{1}$$

follow from the discrete Lagrange-d'Alembert principle, where $D_\bullet L_d$ is the slot derivative with respect to the $\bullet$-th argument and $\mathbf{f}_d$ are the discrete forces. The discrete momenta are given by the discrete Legendre transformation as $\mathbf{p}_n^- = -D_1 L_d(\mathbf{q}_n, \mathbf{q}_{n+1}) - \mathbf{f}_d^-(\mathbf{q}_n, \mathbf{q}_{n+1}, \mathbf{u}_n)$ and $\mathbf{p}_n^+ = D_2 L_d(\mathbf{q}_{n-1}, \mathbf{q}_n) + \mathbf{f}_d^+(\mathbf{q}_{n-1}, \mathbf{q}_n, \mathbf{u}_n)$, where $\mathbf{p}_0^-$ is used for the fist time step.

1.2 Discrete optimal control problem

The DMOC method deals with the problem of finding the discrete control forces $\mathbf{u}_d = \{\mathbf{u}_n\}_{n=0}^{N-1}$ with respect to a – in terms of discrete Euler-Lagrange equations – given system such that a certain discrete objective function J_d or, respectively, a discrete cost function C_d is minimized, i.e.

$$\min_{\mathbf{q}_d, \mathbf{u}_d} J_d(\mathbf{q}_d, \mathbf{u}_d, h) = \min_{\mathbf{q}_d, \mathbf{u}_d, h} \sum_{n=0}^{N-1} C_d(\mathbf{q}_n, \mathbf{q}_{n+1}, \mathbf{u}_n, h) \qquad \text{subject to} \begin{array}{l} \cdot \text{ equation (1)} \\ \cdot \text{ initial and final conditions} \\ \cdot \text{ additional constraints} \end{array} \tag{2}$$

Herein, the infinite dimensional optimal control problem is transcribed into a finite dimensional nonlinear programming problem that can be solved by any standard algorithm, e.g. Sequential Quadratic Programming (SQP).

2 Implementation for the slot car racer

Assuming that the considered slot car has an idealized DC motor (see Fig. 1), the discrete path $\mathbf{q}_d = \left\{[Q_n, \varphi_n]^T\right\}_{n=0}^{N}$ comprises the total amount of moving electric charge Q_n, that has passed any point of the motor windings – where φ_n denotes the rotation angle – at each time step. The discrete control parameter $\mathbf{u}_d = \{U_n\}_{n=0}^{N-1}$ is reduced to a sequence of driving voltages U_n and the current I_n is defined as flow electric charges over time. In the case of a DC motor, the Lagrangian consists only of the magnetic-field coenergy and the mechanical energy of the motor shaft, such that the discrete Lagrangian reads

$$\begin{aligned} L_d(\mathbf{q}_n, \mathbf{q}_{n+1}) = \frac{h}{2} \Bigg\{ \frac{L_a}{h^2} \left(Q_{n+1} - Q_n\right)^2 + \frac{K}{2h} \left(\varphi_{n+1} + \varphi_n\right) \left(Q_{n+1} - Q_n\right) \\ + \frac{\theta}{h^2} \left(\varphi_{n+1} - \varphi_n\right)^2 \Bigg\} \end{aligned} \tag{3}$$

where Θ denotes the inertia of the rigid motor shaft, L_a is the inductance of the windings and K is a machine constant. The discrete forces (with arguments as in equation (1)) are given by

$$\mathbf{f}_d^- = \left[\begin{array}{c} h\, U_n + \frac{R_a}{2} \left(Q_{n+1} - Q_n\right) \\ \frac{h}{4} \left(M_{n+1} + M_n\right) \end{array} \right] \qquad \mathbf{f}_d^+ = \left[\begin{array}{c} h\, U_n - \frac{R_a}{2} \left(Q_n - Q_{n-1}\right) \\ \frac{h}{4} \left(M_n + M_{n-1}\right) \end{array} \right] \tag{4}$$

with the external friction torque $M_n = \frac{r}{i}(F_c \tanh(v_n) + \tau_v v_n)$ and the power dissipation $-R_a I_n$. Herein, a continuous velocity-based friction model – with sliding friction F_c and the viscous friction parameter τ_v – approximates the friction force acting of the slot car. The current is written as a finite difference $I_n = \frac{(Q_{n+1}-Q_n)}{h}$. Under the assumption of a slip-less rolling tire with radius r and the gear ratio of the slot car i, we can compute the velocity $v_n = \frac{r}{i}\frac{\varphi_{n+1}-\varphi_n}{h}$ and the covered distance $s_{n+1} = s_n + v_n h$ of the vehicle (Fig. 2). For this electro-mechanically coupled system the general momenta $\mathbf{p}_n = [p_n^Q, p_n^\varphi]^T$ consist of the flux linkage p_n^Q and the mechanical momentum p_n^φ at each time step.

2.1 Objective function and additional constraints

The time-minimal path can be modeled using different cost functions, were the problem of minimizing the lap time is equivalent to maximizing the velocity – or momentum – for each lap. Within this work, we concentrate on a combined objective function J_d that minimizes the lap time – which corresponds to the sum of time steps – together with the change of driving voltages.

$$J_d(\mathbf{q}_d, \mathbf{u}_d, h) = c_u \sum_{n=0}^{N-1} \left(\frac{U_{n+1} - U_n}{h}\right)^2 + \sum_{n=0}^{N} h \tag{5}$$

Herein, the weighting factor $c_u \in \mathbb{R}$ ensures that the influence of lap time and driving voltages on the cost function are of the same order of magnitude. Furthermore, we can substitute the sum of time steps J_t with the negative sum of the quadratic velocities J_v or the negative sum of the quadratic momenta J_p.

$$J_t(h) = \sum_{n=0}^{N} h \qquad J_v(\mathbf{q}_d, h) = -\sum_{n=0}^{N-1} \left(\frac{s_{n+1} - s_n}{h}\right)^2 \qquad J_p(\mathbf{q}_d, h) = -\sum_{n=0}^{N-1} \left(p_n^{\varphi-}\right)^2 \tag{6}$$

To prevent the slot car from flying off the track, constraints limit the maximal admissible velocities (7) for the track. Here, 14 segments (right curves, left curves and straight elements)

$$v_n \leq \begin{cases} \tilde{v}_1 & \text{for} \quad \tilde{s}_0 \leq s < \tilde{s}_1 \\ \tilde{v}_2 & \text{for} \quad \tilde{s}_1 \leq s < \tilde{s}_2 \\ \vdots & \qquad\qquad \vdots \\ \tilde{v}_{14} & \text{for} \quad \tilde{s}_{13} \leq s \leq \tilde{s}_{14} \end{cases} \tag{7}$$

$$\begin{bmatrix} -\infty \\ \mathbf{s}_0 \\ \mathbf{U}_{min} \\ h_{min} \end{bmatrix} \leq \begin{bmatrix} \mathbf{Q} \\ \mathbf{s} \\ \mathbf{U} \\ h \end{bmatrix} \leq \begin{bmatrix} \infty \\ \mathbf{s}_N \\ \mathbf{U}_{max} \\ h_{max} \end{bmatrix} \tag{8}$$

with length $\tilde{s}_i$ form the track, for which maximal admissible velocities $\tilde{v}_i$ are determined with the camera tracking system. Bounds for the simulation variables (8) guarantee meaningful results. Further constraints define the start position s_0 and the final position s_N.

2.2 Numerical example

As a numerical example, we show the fastest admissible lap, where the vehicle starts with zero acceleration and crosses the finish line with maximal velocity. Here, the parameters $R_a = 4\,\Omega$, $L_a = 2 \times 10^{-2}\,\text{H}$, $K = 4 \times 10^{-3}\,\text{Nm/(radA)}$, $F_c = 2.74 \times 10^{-1}\,\text{N}$ and $\tau_v = 7.81 \times 10^{-1}\,\text{Ns/m}$ are

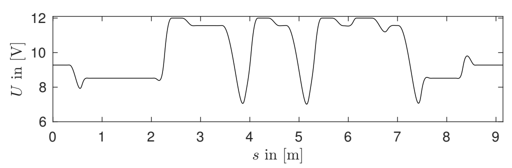

Figure 3. Driving voltage versus covered distance

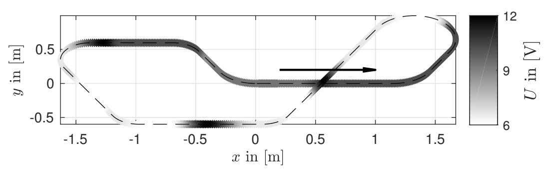

Figure 4. Driving voltage versus track position

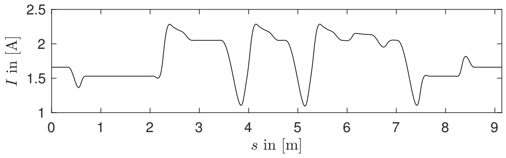

Figure 5. Motor current versus covered distance

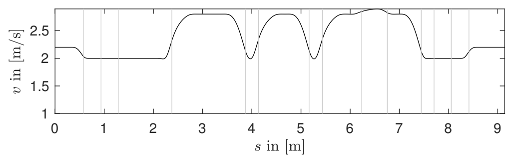

Figure 6. Vehicle velocity versus covered distance

used. The track-length is 9 m, the maximum operating voltage of the DC motor is 12 V and the direction of travel is shown in Fig. 4. The calculated optimal driving voltage for the time minimal path respecting the maximal admissible velocity for the race track are shown in Fig. 3 and Fig. 4. The experimental setup makes it possible to apply this voltage profiles directly to the slot car racer via a computer. Fig. 5 shows the associated motor current and Fig. 6 the velocity of the slot car including the segmentation of the track. As initial guess, a forward dynamics simulation is used. Minimizing the lap time – by including the time step h as an optimization variable in addition to $\mathbf{q}_d$ and $\mathbf{u}_d$ – with the *fmincon* solver from Matlab yields the same result for all three objective functions except of a small numerical error.

Conclusions

This work covers the numerical solution of an optimal control problem for a slot car racer. We investigate several objective functions to minimize the lap time, which influences only the computational effort. Table 1 shows the necessary computational time and iterations to solve the nonlinear programming problem. Obviously, using the approximation for the velocity in J_v influences the computational time and iterations in a negative way. Using the conjugate momentum instead – i.e. the actual momentum at the n-th time node – is physically motivated and also yields the lowest number of iterations. The objective function J_t serves as comparative quantity.

	computational time	iterations	lap time
$J_t(\mathbf{q}_d)$	$745\,s$	297	$3.84\,s$
$J_v(\mathbf{q}_d)$	$2012\,s$	525	$3.84\,s$
$J_p(\mathbf{q}_d)$	$217\,s$	96	$3.84\,s$

Table 1. Computational time and iterations

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