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A Nitsche's method for finite deformation thermo-mechanical contact

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Micro Abstract
This talk presents an extension of Nitsche's method to finite deformation thermo-mechanical contact problems. Besides the coupling of temperature and stress response in the bulk continuum, special focus is put on the consistent enforcement of all involved interface constraints: normal contact, Coulomb's law of friction, heat conduction across the interface and frictional work converted to heat. A set of numerical examples will be presented demonstrating the accuracy of the presented method.

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Introduction

Despite having been an active field of research for many years, computational contact mechanics remains challenging and various methods have been proposed to incorporate contact constraints into finite element analysis. One main difficulty thereby stems from the inequality nature of the contact constraints, both in normal direction (no penetration) and in tangential direction (friction). In many technical applications, also thermal effects at the contact interface, i.e. heat conduction and frictional heating, need to be accounted for. In this work, all interface effects of thermo-mechanical contact will be treated using Nitsche's method. Compared to other numerical methods for contact problems, Nitsche's method has the advantage that, as in penalty methods, no additional unknowns are introduced. However, in stark contrast to penalty methods, Nitsche's method adds penalty terms *consistently* and therefore optimal convergence properties can be achieved for finite penalty parameters.

1 Nitsche's Method for Isothermal Finite Deformation Contact Mechanics

Originally introduced for the weak imposition of boundary conditions [6], a first application of Nitsche's method to contact mechanics was proposed in [7]. A mathematical analysis of the small deformation case has been presented by Chouly et al., see e.g. [3] and extended to frictional effects in [2]. A first extension to finite deformation hyperelasticity can be found in [5].

Let us consider the contact problem of two elastic bodies $\Omega^{(1)}$ and $\Omega^{(2)}$ with finite deformations as depicted in Figure 1. The deformation of the bodies is then governed by the balance equations, which, in their strong form, read

$$\begin{aligned} \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{F}\boldsymbol{S}) + \hat{\boldsymbol{b}}_0 &= \rho_0 \ddot{\boldsymbol{u}} \text{ in } \Omega^{(i)}, \\ (\boldsymbol{F}\boldsymbol{S})\boldsymbol{N} &= \hat{\boldsymbol{t}}_0 \text{ on } \Gamma_\sigma^{(i)}, \\ \boldsymbol{\sigma}\boldsymbol{n} &= \boldsymbol{t}_c^{(i)} \text{ on } \Gamma_c^{(i)}, \\ \boldsymbol{u} &= \hat{\boldsymbol{u}} \text{ on } \Gamma_u^{(i)}, \end{aligned} \tag{1}$$

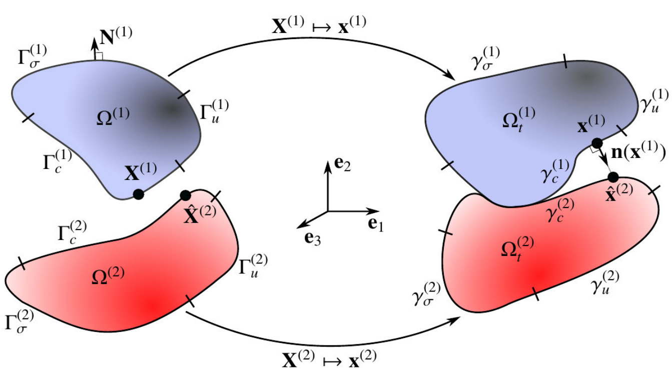


Figure 1. Notation of a two body contact problem.

with appropriate initial conditions and a hyperelastic constitutive law to determine the second Piola–Kirchhoff stress from the right Cauchy–Green tensor $\boldsymbol{C}$ via $\boldsymbol{S} = \partial\psi/\partial\boldsymbol{C}$. At the potential contact boundary γ_c (in the deformed configuration), the balance of linear momentum yields $\boldsymbol{t}_c^{(1)} = -\boldsymbol{t}_c^{(2)}$. We decompose $\boldsymbol{t}_c := \boldsymbol{t}_c^{(1)} = \boldsymbol{\sigma}^{(1)}\boldsymbol{n}$ into its normal part p_n and the tangential traction $\boldsymbol{t}_\tau$. Then, the following inequalities enforce the non-penetration condition and Coulomb friction

$$g_n \geq 0, \;\; p_n \leq 0, \;\; p_n g_n = 0 \;\text{ on } \gamma_c \;\;, \tag{2}$$

$$f^{fr} := \|\boldsymbol{t}_\tau\| - \mu|p_n| \leq 0, \;\; \Delta\boldsymbol{u}_\tau + \beta\boldsymbol{t}_\tau = \boldsymbol{0}, \;\; \beta \geq 0, \;\; f^{fr}\beta = 0 \;\text{ on } \gamma_c \;\;, \tag{3}$$

where g_n represents the normal gap and $\Delta\boldsymbol{u}_\tau$ the relative tangential slip within a time-step. These inequality constraints can be reformulated as equality conditions using complementarity or constraint functions C_n and $\boldsymbol{C}_\tau$ and two penalty parameters γ_n and γ_τ, see e.g. [5]:

$$C_n := p_n - \min[0, p_n + \gamma_n g_n] = 0 \;\;, \tag{4}$$

$$\boldsymbol{C}_\tau := \boldsymbol{t}_\tau - \min\left(1, \frac{-\mu \min[0, p_n + \gamma_n g_n]}{\|\boldsymbol{t}_\tau - \gamma_\tau\Delta\boldsymbol{u}_\tau\|}\right)(\boldsymbol{t}_\tau - \gamma_\tau\Delta\boldsymbol{u}_\tau) = \boldsymbol{0} \;\;. \tag{5}$$

Nitsche's method for contact problems can now be derived by transferring (1) to its variational form and adding terms to impose (4) and (5) weakly. We obtain the method proposed in [5]:

$$\begin{aligned}&\int_\Omega \delta\boldsymbol{u}\rho_0\ddot{\boldsymbol{u}}\,\mathrm{d}\Omega + \int_\Omega (\boldsymbol{F}\boldsymbol{S}) : \nabla_{\boldsymbol{X}}\delta\boldsymbol{u},\mathrm{d}\Omega - \int_\Omega \hat{\boldsymbol{b}}_0\delta\boldsymbol{u}\,\mathrm{d}\Omega - \int_{\Gamma_\sigma} (\boldsymbol{F}\boldsymbol{S})\boldsymbol{N}\cdot\delta\boldsymbol{u}\,\mathrm{d}\Gamma - \int_{\gamma_c^{(1)}} \boldsymbol{t}_c \cdot [\![\delta\boldsymbol{u}]\!]\,\mathrm{d}\gamma \\ &- \int_{\gamma_c^{(1)}} (C_n\boldsymbol{n} + \boldsymbol{C}_\tau)[\![\delta\boldsymbol{u}]\!]\,\mathrm{d}\gamma - \theta_s \int_{\gamma_c^{(1)}} (C_n\boldsymbol{n} + \boldsymbol{C}_\tau)\cdot\mathcal{D}\boldsymbol{t}_c[\delta\boldsymbol{u}]\,\mathrm{d}\gamma = 0 \quad \forall\delta\boldsymbol{u} \;\;,\end{aligned} \tag{6}$$

where $\mathcal{D}\boldsymbol{t}_c[\delta\boldsymbol{u}]$ denotes the directional derivative of $\boldsymbol{t}_c$ in direction of $\delta\boldsymbol{u}$ and $[\![\cdot]\!] = (\cdot)^{(1)} - (\cdot)^{(2)}$ the jump across the contact interface. The first line therein represents the standard weak form of the initial boundary value problem (1) and the second line consistently imposes the contact constraints in a weak manner. For linearized kinematics, the frictionless case of this formulation has been analyzed mathematically in [3] and including Tresca friction in [2]. Stability and optimal convergence rates can be achieved if the penalty parameters are scaled correctly by the mesh size h and the stiffness E of the bodies, i.e. $\gamma_{\{n,\tau\}} = \frac{E}{h}\gamma_{\{n,\tau\},0}$. The parameter θ_s allows to switch between different variants of Nitsche's method: $\theta_s = 1$ gives a symmetric formulation, $\theta_s = 0$ requires less terms (especially avoids the linearized constitutive law $\mathcal{D}\boldsymbol{t}_c[\delta\boldsymbol{u}]$) and $\theta_s = -1$ yields a skew-symmetric formulation, which is shown to be stable for any penalty parameters greater than zero, whereas $\theta_s = 1$ and $\theta_s = 0$ exhibit lower bounds for the penalty parameters.

2 Nitsche's Method for Thermo-Mechanical Contact Problems

When extending Nitsche's method from isothermal to thermo-mechanical contact problems, the treatment of normal and frictional contact constraints presented in the previous section remain virtually unchanged. Only some temperature dependencies might have to be added in the hyperelastic constitutive relation and a potential temperature dependency of the coefficient of friction. Since these extensions are rather straight-forward, this section will focus solely on heat conduction within the two bodies and across the contact interface. The evolution of the temperature T is derived from the balance of energy and, in its weak form, reads

$$\begin{aligned}&\int_\Omega \delta T\rho_0 C_v\dot{T}\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{Q}\cdot\nabla_{\boldsymbol{X}}\delta T\,\mathrm{d}\Omega - \int_\Omega +\hat{R}_0\delta T\,\mathrm{d}\Omega - \int_{\Gamma_q} \hat{Q}_0\delta T\,\mathrm{d}\Gamma \\ &- \int_{\gamma_c^{(1)}} q_c^{(1)}\delta T^{(1)} - q_c^{(2)}\delta T^{(2)}\,\mathrm{d}\gamma = 0 \quad \forall\delta T \in \mathcal{V}_T \;\;,\end{aligned} \tag{7}$$

where $\boldsymbol{Q} = k_0\boldsymbol{C}^{-1}\nabla_{\boldsymbol{X}}T$ denotes the material heat flux according to Fourier's law at finite deformations. The contact heat fluxes $q_c^{(i)}$ therein account for two effects: heat conduction across the contact interface due to a temperature difference and frictional work $\mathcal{P}_\tau = \boldsymbol{t}_\tau \cdot \Delta \boldsymbol{u}_\tau$ being converted to heat:

$$q_c^{(1)} = \beta_c p_n [\![T]\!] - \delta_c \mathcal{P}_\tau \ , \quad q_c^{(2)} = \beta_c p_n [\![T]\!] - (1-\delta_c) \mathcal{P}_\tau \ . \tag{8}$$

Two parameters β_c and δ_c are introduced to control the heat conduction and distribution of frictional work. The simplest way to include these thermal effects into the weak form (7) is to directly substitute (8) into (7). A drawback of this approach arises, however, when β_c becomes large, i.e. there is little resistance to the heat conduction across the interface. In such cases, the terms involving β_c in the weak form become large (compared to the others) and therefore yield an ill-conditioned system. To overcome this issue a Nitsche method is presented in the following which is based on the idea of [4] to deal with general boundary conditions and extended in [1] to interface conditions. Similar to the structural case, the thermal interface condition (8) is added weakly to the variational form (7) by introducing a consistent penalty term with the penalty parameter $\gamma_\vartheta > 0$:

$$\begin{aligned}
&\int_\Omega \delta T \rho_0 C_v \dot{T} \, \mathrm{d}\Omega + \int_\Omega \boldsymbol{Q} \cdot \nabla_{\boldsymbol{X}} \delta T \, \mathrm{d}\Omega - \int_\Omega \hat{R}_0 \delta T \, \mathrm{d}\Omega - \int_{\Gamma_q} \hat{Q}_0 \delta T \, \mathrm{d}\Gamma \\
&- \int_{\gamma_c^{(1)}} \frac{\beta_c p_n}{\beta_c p_n - \gamma_\vartheta} \left\{ q_c(T) \right\}_{1-\delta_c} [\![\delta T]\!] \, \mathrm{d}\gamma + \int_{\gamma_c^{(1)}} \frac{\gamma_\vartheta \beta_c p_n}{\beta_c p_n - \gamma_\vartheta} [\![T]\!] [\![\delta T]\!] \, \mathrm{d}\gamma \\
&+ \theta_\vartheta \int_{\gamma_c^{(1)}} \frac{1}{\beta_c p_n - \gamma_\vartheta} \left\{ q_c(T) \right\}_{1-\delta_c} \left\{ q_c(\delta T) \right\}_{1-\delta_c} \mathrm{d}\gamma - \theta_\vartheta \int_{\gamma_c^{(1)}} \frac{\beta_c p_n}{\beta_c p_n - \gamma_\vartheta} [\![T]\!] \left\{ q_c(\delta T) \right\}_{1-\delta_c} \mathrm{d}\gamma \\
&- \int_{\gamma_c^{(1)}} \mathcal{P}_\tau \left\{ \delta T \right\}_{\delta_c} \mathrm{d}\gamma = 0 \qquad \forall \delta T \in \mathcal{V}_T \ ,
\end{aligned} \tag{9}$$

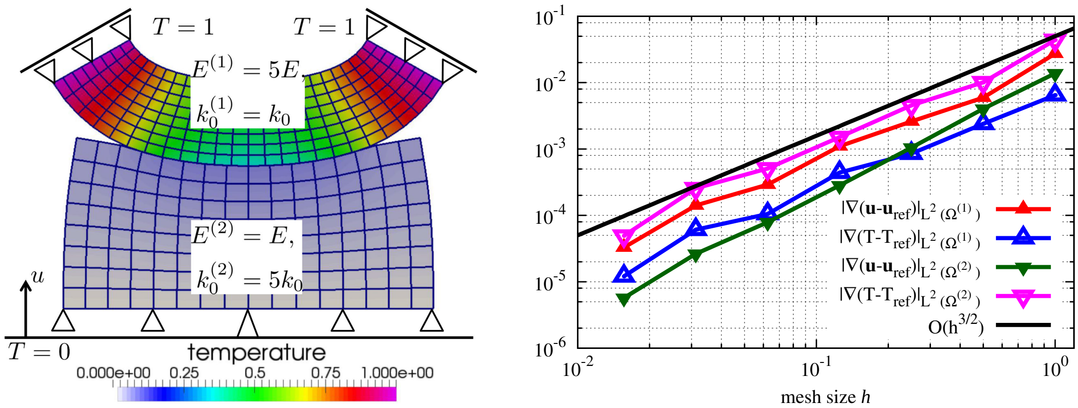
where $q_c^{(i)}(T) = -\frac{k_0^{(i)}}{\det \boldsymbol{F}^{(i)}} \nabla_{\boldsymbol{x}} T^{(i)} \cdot \boldsymbol{n}$ denotes the spatial heat flux derived from Fourier's law and $\left\{ \cdot \right\}_w = w(\cdot)^{(1)} + (1-w)(\cdot)^{(2)}$ the weighted average across the contact interface. Consistency of (9) with (7) can easily be proven by inserting (8) and some algebraic reformulations. Again, stability and optimal convergence can be proven [4], if the penalty parameter is correctly scaled with the mesh size h and the thermal conductivity k_0, i.e. $\gamma_\vartheta = \frac{k_0}{h} \gamma_{\vartheta,0}$. The parameter θ_ϑ again allows to switch between different variants. Analogous to the structural problem, $\theta_\vartheta = 1$ yields a symmetric system and $\theta_\vartheta = 0$ involves less terms. Both those variants exhibit a lower bound for the penalty parameter. Finally, $\theta_\vartheta = -1$ yields a skew-symmetric Nitsche method stable for any penalty parameter $\gamma_\vartheta > 0$. Unlike the substitution method described above, the Nitsche method (9) remains well-conditioned, also in the limit cases $\beta_c \to 0$ and $\beta_c \to \infty$.

3 Numerical Example

As numerical example, we demonstrate the convergence behavior of the proposed method within a 2-dimensional setup, where a rectangular block ($\Omega^{(2)}$) is pressed against an initially circular arc ($\Omega^{(1)}$). Figure 2(a) illustrates the boundary conditions as well as the deformed configuration and temperature distribution at a steady-state. Using $\mathcal{Q}^2$ elements and uniform mesh refinement, an exemplary convergence behavior of H^1-semi-norms of displacements and temperatures on the two sub-domains (compared to a numerical reference solution) is given in Figure 2(b). Here, the symmetric Nitsche method has been used for both contact and heat conduction, i.e. $\theta_s = \theta_\vartheta = 1$ with $\gamma_{n,0} = \gamma_{\vartheta,0} = 2$. As usual in computational contact mechanics, convergence rates with uniform mesh refinement are limited by the regularity of the solution rather than the approximation order, such that the observed order $\mathcal{O}(h^{3/2})$ is considered optimal. Similar convergence behavior is also observed for other combinations of the proposed methods.

4 Conclusions

This contribution presents the application of Nitsche's method to finite deformation thermo-mechanical contact. Therefore, the isothermal hyperelastic formulation proposed in [5] is



(a) Problem setup

(b) convergence

Figure 2. Thermo-mechanical contact – Exemplary solution for mesh size $h = 1/8$ and convergence behavior.

extended to thermo-elasticity and a temperature dependent friction coefficient. Moreover, heat conduction across the contact interface as well as frictional heating is accounted for. The simple substitution method for the thermal interface condition becomes ill-conditioned for low thermal contact resistances, whereas the presented Nitsche's method for the thermal interface effects is well-conditioned over the entire range of interface parameters. Owing to the consistency of the method, optimal convergence behavior of the finite deformation problem can be achieved. Contrasting other variationally consistent discretization approaches for contact problems such as the mortar method, Nitsche's method does not require additional degrees of freedom (Lagrange multipliers), but remains a purely primal formulation.

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