**Proceedings of the 7th GACM Colloquium on Computational Mechanics
for Young Scientists from Academia and Industry
October 11-13, 2017 in Stuttgart, Germany**

# On the shape functions for the contact pressure in mortar methods

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**Micro Abstract**
Mortar formulations differ on the choice of shape functions for approximation of the contact pressure. The shape functions can be identical to the standard, weighted standard, or the dual shape functions. In this contribution, we will unify all the above choices by starting with a least-squares condition. That is, the shape functions are constructed such that the smoothed contact pressure fits best to the raw contact pressure. Various other choices are also compared and discussed.

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## Introduction

This article presents a construction procedure for various shape functions that are used for approximation of the contact pressure in mortar methods. Consider a pair of discretized slave and master surfaces. A point $\boldsymbol{x}$ on the discretized slave surface $\Gamma^{\mathrm{s}} = \cup_{e=1}^{n_{\mathrm{el}}} \Gamma^{\mathrm{s}}_e$ can be interpolated from nodal values $\mathbf{x}_A$ by the standard shape function $N_A$ as $\boldsymbol{x} = \sum_{A=1}^{n} N_A \mathbf{x}_A$,[1] where $n$ is the total number of nodes of $\Gamma^{\mathrm{s}}$. In mortar methods, the impenetration constraints are enforced in a weak sense by the potential $p_A\, g_{\mathrm{n}A} = 0$, where $p_A$ and $g_{\mathrm{n}A}$ denote the weighted contact pressure and normal contact gap collocated at node $A$, respectively [3]. Consider the penalty method, $p_A$ and $g_{\mathrm{n}A}$ can be defined by

$$p_A := \epsilon\, g_{\mathrm{n}A} = \int_{\Gamma^{\mathrm{s}}_A} M_A\, p \;\, \mathrm{d}A \;, \quad p := \epsilon g_{\mathrm{n}} \;, \tag{1}$$

where $\Gamma^{\mathrm{s}}_A$, $\epsilon$, $p$, and $g_{\mathrm{n}}$ denote the nodal support domain, the penalty parameter, the (point-wise) raw contact pressure and normal gap, respectively; $M_A$ denotes the mortar shape function for the mortar contact pressure, defined by $p^{\star} := M_A\, p_A$. It should be distinguished from the smoothed contact pressure, defined by $\hat{p} := N_A\, p_A$. In the following, we present a least-squares approach for the construction of $M_A$.

## 1 Various shape functions for the contact pressure

**1.1. Global least-squares shape function (M-GLS)** In this approach, the shape function $M_A$ is constructed such that the smoothed contact pressure fits best to the raw contact pressure. This can be done by minimizing the global least-squares functional

$$\Pi_{\mathrm{LS}} := \int_{\Gamma^{\mathrm{s}}} \frac{1}{2}(\hat{p} - p)^2 \;\, \mathrm{d}A \;. \tag{2}$$

Here, the integration domain $\Gamma^{\mathrm{s}}$ can be in either the current configuration or a reference configuration (e.g. the initial or the previous-load-step configuration). The later choice would significantly simplify computation of $M_A$. From $\delta\Pi_{\mathrm{LS}} = 0$ follows that

[1] In the following, the Einstein summation convention for indices ($A$, $B$, $C$, ..) is adopted unless otherwise stated

$$p_A = \int_{\Gamma^s} [L_{AB}]^{-1} \, N_B \, p \; \mathrm{d}A \; , \tag{3}$$

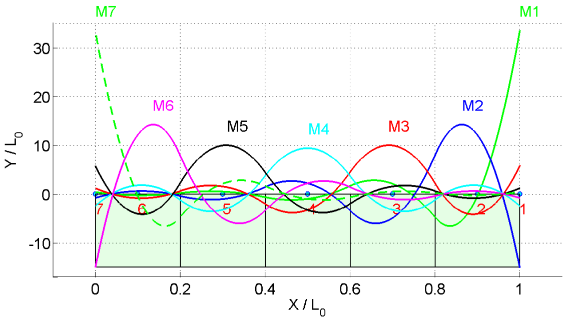
where $L_{AB}$ is a (symmetric) mass matrix defined by

$$L_{AB} := \int_{\Gamma^s} N_A \, N_B \; \mathrm{d}A \; . \tag{4}$$

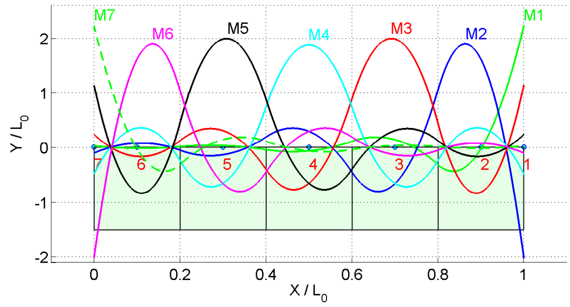
By comparing Eq. (3) with Eq. (1), we obtain

$$M_A := N_B \, [L_{BA}]^{-1} \; . \tag{5}$$

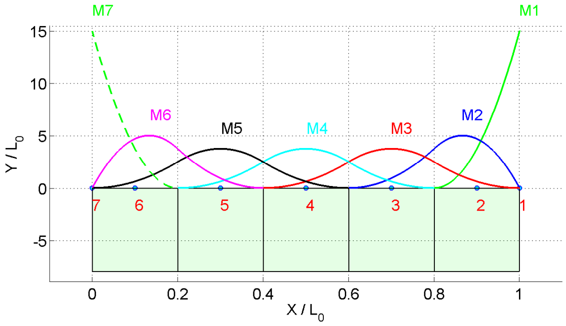
Since these mortar shape functions are derived from a global least squares approach, we denote them M-GLS. These functions are plotted in Fig. 1a. Note that, $L_{AB}$ is a dense matrix and the support area of $M_A$ spans the entire slave surface. Besides, $M_A$ does not satisfy partition of unity (PU), i.e. $\sum_A M_A \neq 1$. This issue will be addressed next.



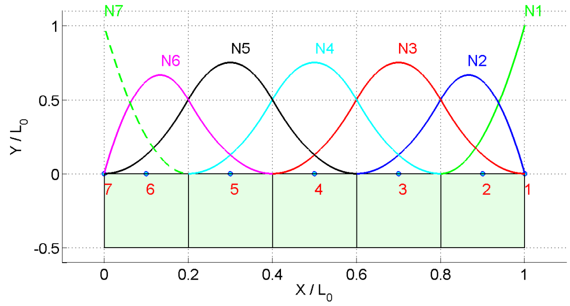

a. Global least-squares (M-GLS)



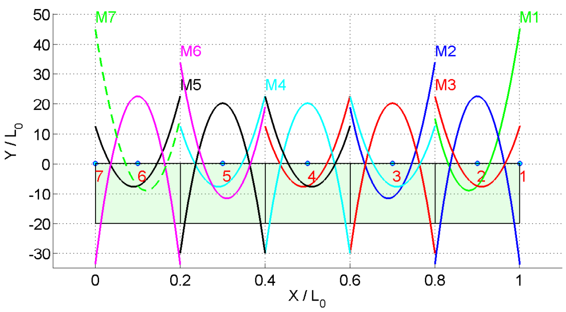

b. PU Global least-squares (M-GLS*)



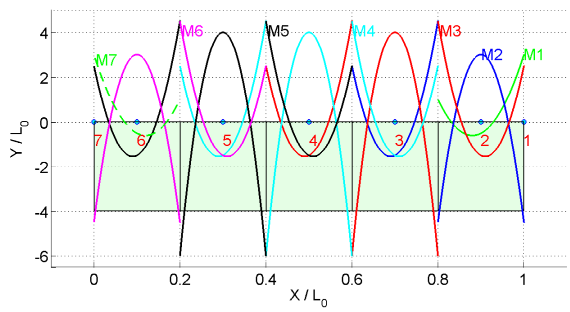

c. Lumped least-squares (M-LmLS)




d. PU lumped least-squares (M-LmLS*)




e. Local least-squares (M-LcLS)




f. PU local least-squares (M-LcLS*)

Figure 1. Various mortar shape functions $M_A$ for one-dimensional quadratic NURBS.

**1.2. Global least-squares with partition of unity (M-GLS*)** In order to satisfy partition of unity (PU), a normalization technique can be applied at the nodes as

$$M_A := N_B \, [L_{BC}]^{-1} \, W_{CA} \; , \tag{6}$$

where $W_{CA}$ is computed from a lumped mass of $L_{AB}$ as

$$W_{CA} := \delta_{CA} \int_{\Gamma^{\mathrm{s}}} N_A^{\mathrm{s}} \, \mathrm{d}A \ , \quad \text{(no sum over A)} \ . \tag{7}$$

Here $\delta$ denotes the Kronecker delta function. The resulting shape function (6) is thus denoted by M-GLS* and depicted in Fig. 1b.[2] However, the area support of the nodes is the same as M-GLS. Note that M-GLS* can alternatively be obtained by the so-called global biorthogonality condition [2, 4].

### 1.3. Lumped least-squares (M-LmLS)

If the mass matrix $[L_{AB}]$ in M-GLS (5) is approximated by the lumped matrix (7), we obtain a shape function $M_A$ that spans locally (element-wise) as

$$M_A := N_B \, W_{BA}^{-1} \ . \tag{8}$$

It is thus referred to as M-LmLS shape function and plotted in Fig. 1c. Note that this shape function is equivalent to the weighted standard shape function [1].

**1.4. Lumped least-squares with partition of unity (M-LmLS*)** Similarly, the nodal normalization can be applied to M-LmLS shape function (8) to satisfy partition of unity. The resulting shape function is thus denoted by M-LmLS*. In this case, $M_A$ becomes identical to the standard shape function (see Fig. 1d), i.e.

$$M_A \, := \, N_A \ . \tag{9}$$

This shape function has been considered in the mortar formulation of Puso and Laursen [3].

### 1.5. Local least-squares with partition of unity (M-LcLS*)

Now, instead of defining the least-squares functional globally as is done in Eq. (2), we can state the least-squares condition over each FE element $e$ separately,

$$\Pi_{\mathrm{LS}}^e := \int_{\Gamma_e^{\mathrm{s}}} \frac{1}{2} (\hat{p} - p)^2 \ \mathrm{d}A_e \quad \forall e = 1, \ \ldots, \ n_{\mathrm{el}} \ . \tag{10}$$

In analogy to Eq. (5), we get

$$M_A^e := N_B \, [L_{BA}^e]^{-1} \ , \quad \text{with} \quad L_{AB}^e := \int_{\Gamma_e^{\mathrm{s}}} N_A \, N_B \ \mathrm{d}A \ . \tag{11}$$

Next, an additional averaging technique is required for smoothing unequal $\hat{p}$ at nodes. Using the nodal normalization technique results in

$$M_A := N_B \, [L_{BC}^e]^{-1} W_{CA}^e \ , \quad \text{with} \quad W_{CA}^e := \delta_{CA} \int_{\Gamma_e^{\mathrm{s}}} N_A^{\mathrm{s}} \ \mathrm{d}A \ , \quad \text{(no sum over } A) \ . \tag{12}$$

Note, that $W_{CA}^e$ is defined locally and should not be confused with $W_{CA}$ defined globally in Eq. (7). Since $M_A$ are derived from a local (i.e. elementwise) least squares approach and have the PU property, we denote them M-LcLS*. By design, it has the same support as the standard shape function array $N_A$. However, $M_A$ may have strong discontinuities over the element boundaries as shown in Fig. 1f. Like M-GLS, M-LcLS* shape function (12) can also be constructed from the so-called biorthogonality condition, but now locally [4]. In this regard, M-LcLS* is also called *dual* shape function.

### 1.6. Local least-squares without partition of unity (M-LcLS)

Applying the weighting scheme similar to M-LmLS (8) to M-LcLS* results in a so-called M-LcLS without partition of unity as

$$M_A := N_B \, [L_{BC}^e]^{-1} \, W_{CD}^e \, W_{DA}^{-1} \ . \tag{13}$$

It is plotted in Fig. 1e. A comparison of M-LcLS with its counterpart M-LcLS* will be discussed in the following.

[2] where (*) indicates the version satisfying partition of unity

## 2 On choosing the shape functions for the contact pressure

Fig. 2 shows the contact pressures for the interpenetration of a rigid sphere and a flat surface described by quadratic NURBS (without any contact enforcement). As shown, non-PU shape functions (i.e. M-GLS, M-LmLS, and M-LcLS) yield higher mortar pressures than PU counterparts. Therefore, using the non-PU shape functions for mortar methods would usually require less value of the penalty parameter than the corresponding PU ones considering the same penetration.

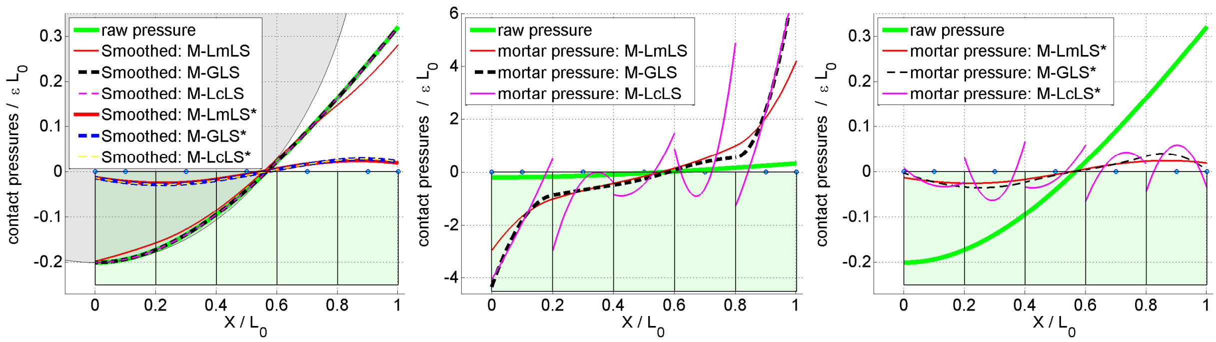



Figure 2. Interpenetration of a sphere and a quadratic NURBS surface: smoothed contact pressures $\hat{p} := N_A\, p_A$ considering various shape functions $M_A$ (left). Corresponding mortar pressures $p^\star := M_A\, p_A$ without PU (middle) and with PU (right) shape functions. The raw pressure $p$ (green line) is the same in all figures.

M-LmLS (likewise M-LmLS*) is simple but least accurate. Meanwhile, M-GLS appears to provide the best fit of $\hat{p}$ to $p$. The nodal support of $M_A$, however, spans the whole contact surface. In contrast, the local support can be obtained for M-LcLS, but its mortar pressure $p^\star$ may have strong discontinuities over element boundaries. Further, since M-LcLS is elementwise constructed, the consistency treatment at the contact boundary may lead to ill-conditioning of the system of equations when the contact boundary approaches element boundaries [2] . This issue does not appear for shape functions that are based on a global least-squares approach (like M-GLS and M-LmLS).

### Conclusion

Various shape functions for the contact pressure for the mortar methods can be constructed based on the least squares approach. With this, the shape functions that are employed by [3], [1], and [2] can be identified as M-LmLS*, M-LmLS, and M-LcLS*, respectively. Main features of different choices have been discussed.

### References

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