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Adaptive finite elements for contact problems based on efficient and reliable residual-type a posteriori estimators

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Micro Abstract
The talk deals with the adaptive numerical simulation of contact problems based on residual-type a posteriori estimators. The estimators are easy to compute and provably reliable, efficient and localized. The latter properties enable a good resolution of the free boundary while avoiding over-refinement in the active set of contact. We consider continuous as well as discontinuous finite elements for the numerical simulation of static and time-dependent contact problems.

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Introduction

Contact problems occur in a wide range of areas, including mechanical engineering or biomechanics. To name a few examples, contact arises in fabrication machines such as grinding or forming tools, in the automotive industry, for example in crash tests and in abrasion of tires, and in biomedical engineering, for example in the stress prediction between cartilage and prothesis.

The colliding bodies deform depending on the material law but do not penetrate each other. The constraint of non-penetration $u_\nu \leq g$ on the potential contact boundary Γ_C, where u_ν is the solution in the direction of constraints ν, and g describes the distance between the two colliding bodies, has to be considered in a mathematically thorough formulation, e.g as saddle point problem or variational inequality. We consider the formulation as variational inequality

$$a(u, v-u) \geq \langle f, v-u \rangle \quad \forall v \in \mathcal{K} \tag{1}$$

where $\mathcal{K} := \{v \in H^1(\Omega) \mid v_\nu \leq g \text{ on } \Gamma_C\}$ is the admissible set and f a given external force. The bilinear form $a(\cdot,\cdot)$ contains the information about the material law of the colliding bodies. For example in linear elasticity $a(\cdot,\cdot) = \int_\Omega \sigma(\cdot)\epsilon(\cdot)$ with the linearized strain tensor ϵ and σ given by Hooke's law. For the discretization in space of (1) we consider continuous as well as discontinuous linear finite elements.

Due to the constraints contact problems are non-smooth problems and thus the numerical simulation to reach a certain accuracy is usually very expensive. Therefore, the adaptive numerical simulation based on a posteriori estimators is in great demand. In the case of linear elliptic boundary value problems from mechanics, the construction of a posteriori error estimators has reached a certain maturity, see [6] for an overview over several kinds of a posteriori estimators. In contrast, in the case of nonlinear and non-smooth problems there are still many issues which remain to be solved.

A popular a posteriori estimator, which appears attractive in view of its simplicity and generality is standard residual estimation. In this talk we present residual-type a posteriori error estimators for contact problems which are equivalent to the error measure, i.e. the estimator is reliable (upper bound to the error) and efficient (lower bound to the error), such that the error is neither overestimated nor underestimated. Especially the proof of efficiency and the localization of the estimator contributions related to the constraints are difficult steps in the proof.

Our theoretical findings are supported by numerical studies for static and time-dependent contact problems.

1 Derivation of residual-type a posteriori estimators for contact problems

In the case of contact problems the basic difficulty in the derivation of residual-type estimators stems from the contact forces which are evoked by the constraint of non-penetration. For linear equations without constraints the equivalence between the error in the energy norm and the dual norm of the residual is the starting point for the derivation of residual estimators. As the dual norm of the residual is not a computable quantity, a computable estimator of the residual is sought-after. Dealing with the bilinear form $a(\cdot,\cdot)$ the linear residual $\langle R_{\mathrm{lin}}(u_{\mathrm{m}}),\cdot\rangle_{-1,1} := f(\cdot) - a(u_{\mathrm{m}},\cdot)$ for the discrete solution $u_{\mathfrak{m}}$ can be defined even for the variational inequality. But in the case of variational inequalities the linear residual is not equivalent to the error. This is due to the fact that in the case of variational inequalities the linear residual consists of two parts, one is related to the error and the other one to the discrete constraining force. Thus, the derivation of residual-type estimators for variational inequalities requires other techniques which take into account the special role and structure of the discrete constraining force which is part of the linear residual.

1.1 Finite element solution of the Signorini problem

We propose and analyze an efficient and reliable residual-type estimator for the Signorini problem [3]. The disturbed relation between the linear residual and the actual error is due to the fact that not only the displacements u but also the contact forces λ are unknowns of the system. Thus, giving consideration to both errors we measure the error by $\|u-u_{\mathfrak{m}}\|_1+\|\lambda-\tilde{\lambda}_{\mathfrak{m}}\|_{-1}$. Further, the role of the residual is taken by the Galerkin functional $\langle \mathcal{G}_{\mathfrak{m}},\cdot\rangle_{-1,1} = a(u-u_{\mathfrak{m}},\cdot)+\left\langle \lambda-\tilde{\lambda}_{\mathfrak{m}},\cdot\right\rangle$. This concept was first used for obstacle problems in the work [5]. Here, the so-called quasi-discrete contact force $\tilde{\lambda}_{\mathfrak{m}}$ is an extension of the discrete contact force

$$\langle \lambda_{\mathfrak{m}},\varphi_{\mathfrak{m}}\rangle_{-1,1} := \langle f,\varphi_{\mathfrak{m}}\rangle - a(u_{\mathfrak{m}},\varphi_{\mathfrak{m}}) \quad \forall \varphi_{\mathfrak{m}} \in H^1_{\mathfrak{m},0} \subset H^1_0.$$

to a functional on H^1, reflecting the properties of the contact force as e.g. sign condition and relation to the $d-1$-dimensional boundary part where the constraints are imposed. The appropriate definition of the quasi-discrete contact force is important for the proofs of upper and lower bounds of the estimators. Distinguishing between areas of full- and semi-contact in which the bodies are completely or partially in contact, cf. [1, 4], we exploit the local structure of the solution as well as of the constraining force and we achieve a localization of the error estimator contributions. In the area of full-contact no estimator contribution related to the nonlinearity appears. Thus, the fact that the solution in direction of the constraints is fixed to the obstacle and cannot be improved, is reflected by the estimator.

The error estimator consists of the standard estimator contributions for linear elliptic equations plus additional terms related to the constraints. Thus, in the case no contact occurs, it reduces to the standard residual estimator. The additional contact-related contributions are

$$\eta_{5,p} := h_p^{\frac{1}{2}} \left\|\hat{\sigma}_\nu(u_{\mathfrak{m}})\right\|_{\gamma_{p,C}}, \quad \forall p \in N^C_{\mathfrak{m}} \backslash N^{\mathsf{fC}}_{\mathfrak{m}} \quad \text{“contact stress residual”}$$

$$\eta_{6,p} := (s_p d_p)^{\frac{1}{2}} \text{ with } d_p := \int_{\tilde{\gamma}_{p,C}} (g_{\mathfrak{m}} - u_{\mathfrak{m},\nu})\phi_p \text{ and } s_p := \frac{\langle \lambda_{\mathfrak{m},\nu},\phi_p\rangle_{-1,1}}{\int_{\gamma_{p,C}} \phi_p} \quad \text{“complementarity residual”}$$

where $\hat{\sigma}_\nu(u_{\mathfrak{m}})$ are the surface stresses at the contact boundary, $N^{\mathsf{fC}}_{\mathfrak{m}}$ denotes the set of all full-contact nodes and $\gamma_{p,C}$ the set of all contact boundary sides sharing the node p.

The proofs of upper and lower bound are given for meshes of simplices in 2D and 3D. Moreover, the case of non-discrete gap functions is addressed and several numerical results show the performance of the estimator for unstructured and mixed meshes.

Amongst others we consider the example where a linear elastic cube comes into contact with a rigid ball. The starting grid consists of hexahedra and the final number of degrees of freedom is $\approx 4 \cdot 10^6$. In Figure (1) we show the adaptively refined grid on the contact surface of the linear elastic cube and projected on the obstacle. Obviously, the critical region between the actual and non-actual contact boundary is well refined while the interior of the contact area is less refined avoiding over-refinement in this region where the two bodies stick to each other.

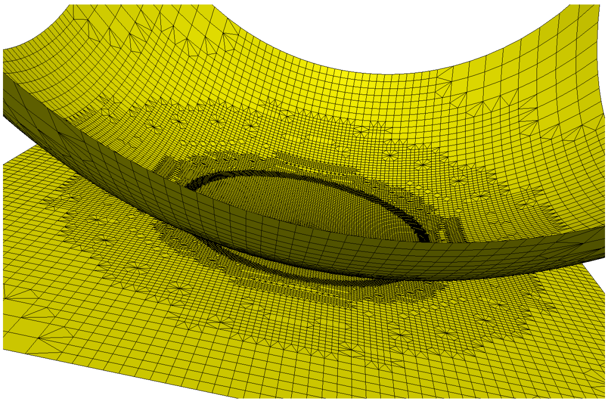


Figure 1. *Adaptively refined grid of the contact surface and projected on the rigid ball*

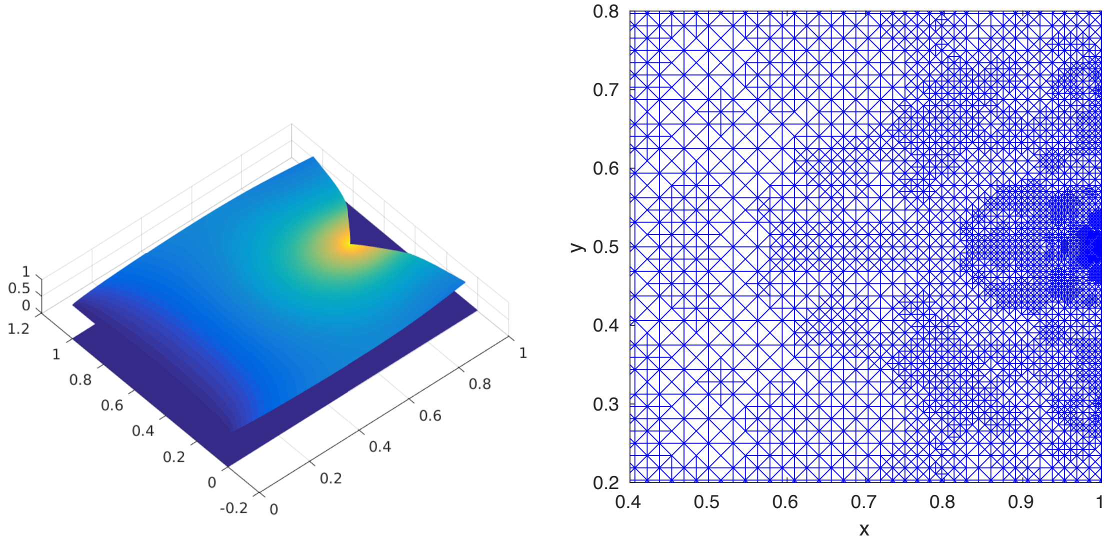
1.2 Discontinuous finite element solution of the Signorini problem

Discontinuous finite elements are more flexible, e.g. for unstructured grids, different polynomial degrees on each element and parallel computing. However, the number of degrees of freedom is higher than for continuous finite elements. Thus, adaptive mesh refinement is an important tool to decrease the computational costs.

We assume that a piecewise linear discontinuous Galerkin finite element discretization has been chosen. As an example, the interior penalty method which is a popular approach of discontinuous Galerkin methods for elliptic equations is considered.

We present a localized, efficient and reliable estimator applicable to two- and three-dimensional contact problems [7]. We transfer the approach of classifying the area of contact in full- and semi-contact zones to discontinuous finite elements, for the first time. Further, we introduce an appropriate definition of the quasi-discrete contact force and we use a subtle splitting of the contributions in the Galerkin functional. Thus, cancellation effects could be maximized so that estimator contributions related to the non-penetration condition vanish in the area of full-contact. This implies that the estimator contributions are localized and overestimation in the area of full-contact is avoided. Moreover, the structure of the discontinuous finite elements allows for a more precise localization of the area of full-contact compared to continuous finite elements.

In Figure 2(a), we see the deformed configuration of a linear elastic square which collides with a wedge as obstacle. The adaptively refined grid in the area $[0.4, 1] \times [0.2, 0.8]$ for this example is shown in Figure 2(b). One can see that only around the corner where the tip of the wedge comes into contact and at the free boundary where the body detaches the grid is higher refined, but not at the whole contact boundary.



(a) *deformation* **(b)** *adaptively refined grid*

Figure 2. Contact with a wedge

Conclusions

Summing up, we present estimators for the continuous as well as discontinuous finite element solutions of contact problems. The estimator contributions are easy to compute and as the estimator reduces to the standard residual estimator if no contact occurs it can be easily added to existing programs for adaptive refinement.

We prove efficiency and reliable for the Signorini problem. However, the method can be used for time-dependent problems as e.g. the dynamic Signorini problem. Using Rothe's method the resulting variational inequality in each time step has a similar structure as for the static Signorini problem. Thus, combined with an appropriate time discretization for contact problems [2] it is possible to detect and resolve the local non-smooth effects at the contact boundary in space and time.

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