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Partial mechanics of far fields in elastoplastic structures

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Micro Abstract
We propose a reduced-modeling protocol that accounts of topological modifications in elastoplastic structures. We assume that topological modifications and mesh adaptations are restricted to a subdomain termed the zone of interest. This zone of interest is surrounded by an hyper-reduced order model that propagates boundary conditions by using empirical modes.

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Introduction

This paper presents a reduced-modeling protocol that accounts of morphological modifications in structures submitted to elastoplastic transformations. We assume that the morphological modifications and the related mesh adaptations are restricted to a subdomain termed the region of interest (ROI). In the proposed protocol, a domain decomposition in two subdomains is performed: one subdomain is the region of interest, the remaining part of the domain is termed the surrounding domain. A recent numerical approach that couples hyper-reduced approximations and finite element approximations, proposed in [2] for the solution of Navier-Stokes equations, is extended to elastoplasticity. Here, we take advantage of the finite-elements versatility to account for the morphological modifications. An hyper-reduced approximation is set up over the surrounding domain. It aims at propagating boundary conditions towards the region of interest. In this subdomain, both approximations interact. The propose protocol enables to develop approximations for fields that surround a region of interest. The equations fulfilled by theses fields are set up on a reduced integration domain, by following a usual hyper-reduction method. Since only few elements of the surrounding domain are involved in the reduced order model, this approach is named partial mechanics of far-fields (PMFF). It is very versatile regarding morphological modifications. Offline and online phases are proposed for the construction and the incorporation of the far-fields approximations. The novelty of this work is the modeling of morphological and nonparametric modifications in plasticity, enabled by an hybrid full-order/reduced-order model.

Hybrid full-order/reduced-order models have been proposed in the literature to circumvent the lack of accuracy of reduced-basis approximations for specific problems. Hybrid modeling has been proposed in [4] for nonlinear problems having nonlinear terms restricted to a subdomain. Here, we do not assume localized nonlinearities. In [1, 5, 6] empirical modes have been coupled to finite element approximations for nonlinear structural problems involving heat conduction or plasticity or damage. Critical damages are known to generate morphological modifications in damage simulation. Here, equilibrium equations in the surrounding domain are setup by the recourse to a reduced integration domain (RID).

The proposed equations sound like Hyper-reduced equations, but attention must be paid to coupling terms in the region of interest in order to strongly couple the local finite element (LFE) approximation and the reduced basis approximation related to far fields.

Matrix and second-order tensors are denoted by a bold capital letter $\mathbf{A}$. Vectors are denoted by a small bold letter $\mathbf{a}$. Entries of a matrix $\mathbf{A}$ are denoted by a_{ij}. The 2-norm for vectors is denoted by $\| \cdot \|_2$. In the remainder, the vector containing the entries selected by a set of row indices denoted by $\mathcal{H}$, reads $\mathbf{r}[\mathcal{H}] \in \mathbb{R}^{\mathrm{Card}(\mathcal{H})}$. The row selection applied to all columns of $\mathbf{V} \in \mathbb{R}^{\mathcal{N} \times N}$ reads $\mathbf{V}[\mathcal{H}, :] \in \mathbb{R}^{\mathrm{Card}(\mathcal{H}) \times N}$. We denote $\mathbf{u}(\cdot, t)$ the displacement field at time t. The usual residual of the finite-element equilibrium-equations is denoted by $\mathbf{r}(\mathbf{u}) \in \mathbb{R}^{\mathcal{N}}$. The finite element shape functions are $(\boldsymbol{\varphi}_i)_{i=1}^{\mathcal{N}}$.

1 Partial mechanics of far fields

Let's assume that a reduced basis is available for the approximation of the displacements over the entire domain, denoted by Ω. This reduced basis is termed the far field reduced basis. The far field modes of this reduced basis are denoted by $\boldsymbol{\psi}_k^R$ such that:

$$\boldsymbol{\psi}_k^R(\mathbf{x}) = \sum_{i=1}^{\mathcal{N}} \boldsymbol{\varphi}_i(\mathbf{x})\, v_{ik}^R, \quad k = 1, \dots, N,\ \mathbf{V}^R \in \mathbb{R}^{\mathcal{N} \times N}, \quad N < \mathcal{N},\ \mathrm{rank}(\mathbf{V}^R) = N \tag{1}$$

In practice, this reduced basis is obtained by the usual proper orthogonal decomposition of solutions generated during an unsupervised machine learning stage. The ROI, denoted by Ω_F, is defined by a given set of degrees of freedom $\mathcal{F}$ where the LFE approximation is setup:

$$\Omega_F = \cup_{i \in \mathcal{F}} \mathrm{supp}(\boldsymbol{\varphi}_i) \tag{2}$$

For the sake of simplicity, degrees of freedom are ordered such that $\mathcal{F} = \{1+\mathcal{N}-\mathrm{Card}(\mathcal{F}), \dots, \mathcal{N}\}$. Then the empirical modes of the hybrid approximation are:

$$\boldsymbol{\psi}_k(\mathbf{x}) = \sum_{i=1}^{\mathcal{N}} \boldsymbol{\varphi}_i(\mathbf{x})\, v_{ik}, \quad k = 1, \dots, N + \mathrm{Card}(\mathcal{F}), \quad \mathbf{V} = \begin{bmatrix} \mathbf{0} & \mathbf{V}^R \\ \mathbf{I} & \end{bmatrix} \tag{3}$$

Here $\mathbf{I} \in \mathbb{R}^{\mathrm{Card}(\mathcal{F}) \times \mathrm{Card}(\mathcal{F})}$ is the identity matrix related to the degrees of freedom in $\mathcal{F}$. Formally, when the mesh on Ω_F is modified to account for morphological changes, the modes in $\mathbf{V}^R$ have to be projected by using the finite element shape functions. In the sequel, we assume that $\mathrm{rank}(\mathbf{V}) = N + \mathrm{Card}(\mathcal{F})$.

Here, an hyper-reduced order model is a boundary value problems restrained to a reduced integration domain [7], denoted by $\widehat{\Omega}$, whose extent is the union of few shape-function supports such that:

$$\widehat{\Omega} = \cup_{i \in \mathcal{H}} \mathrm{supp}(\boldsymbol{\varphi}_i) \tag{4}$$

We adopt the last approach because it preserves the usual assembly loop on the elements involved in finite element simulations. Then all local constitutive equations can be considered, especially elastoplastic models.

When the boundary value problem is restrained to a subdomain $\widehat{\Omega}$ by using an HROM, an additional boundary condition is introduced in order to obtain a well posed problem. The additional boundary condition is set up on the interface between the RID and the remaining part of the domain. This interface reads: $\boldsymbol{\Gamma} = \widehat{\Omega} \cap \widetilde{\Omega}, \quad \widetilde{\Omega} = \overline{\Omega \backslash \widehat{\Omega}}$ where $\overline{\Omega \backslash \widehat{\Omega}}$ is the complement of $\widehat{\Omega}$ extended to its boundary. By following [7], the additional boundary condition is similar to a Dirichlet boundary condition. It requires a proper restriction of the modes to $\widehat{\Omega}$, such that the restrained modes are kinematically admissible fields with respect to an homogeneous Dirichlet boundary condition on $\boldsymbol{\Gamma}$. The convenient restriction of the modes to the RID is denoted by $\widehat{\boldsymbol{\psi}}_k$, such that:

$$\widehat{\boldsymbol{\psi}}_k(\mathbf{x}) = \sum_{i \in \mathcal{L}} \boldsymbol{\varphi}_i(\mathbf{x})\, v_{ik},\ k = 1, \dots, N, \quad \mathcal{L} = \left\{ i \in \{1, \dots, \mathcal{N}\} \,|\, \int_{\widetilde{\Omega}} \boldsymbol{\varphi}_i^2(\mathbf{x})\, dx = 0 \right\} \tag{5}$$

where $\mathcal{L}$ is the set of degrees of freedom that are not on the interface $\mathbf{\Gamma}$ and neither in the outer part of the RID. By construction we have: $\mathcal{H} \subseteq \mathcal{L}$. The hyper-reduced equilibrium equations read:

$$\mathbf{V}[\mathcal{L}, :]^T\, \mathbf{r}(\mathbf{u})[\mathcal{L}] = 0, \text{with } \mathbf{u} = \sum_{k=1}^{N+\text{Card}(\mathcal{F})} \boldsymbol{\psi}_k(\mathbf{x})\, \gamma_k(t) \tag{6}$$

By following the HR method proposed in [7], $\mathcal{H}$ contains the interpolation indices generated by the discrete empirical interpolation method (DEIM) [3] applied to $\mathbf{V}$. There is various way to extend $\mathcal{H}$ in order to include more indices. More details about this extension are given in [7]. Here, interpolation indices are also extracted from a POD basis related to stress. These indices are then connected to indices of degrees of freedom, through the mesh, and included in $\mathcal{H}$. Here, because of the shape of the hybrid matrix $\mathbf{V}$, the first interpolation indices computed by the DEIM are all the indices in $\mathcal{F}$. Then the RID contains the ROI and $\mathcal{L} = \mathcal{L}^R \cup \mathcal{F}$ with $\mathcal{L}^R \cap \mathcal{F} = \emptyset$.

2 Coupling terms

The domain Ω is split into Ω_F and the remaining part Ω_R. The interface between Ω_R and Ω_F is denoted by Φ, $\Phi = \Omega_F \cap \Omega_R$. The set of degrees of freedom on Φ is denoted by $\mathcal{I}$:

$$\mathcal{I} = \left\{ i \in \{1, \ldots, \mathcal{N}\} \mid \int_\Phi \varphi_i^2(\mathbf{x})\, ds > 0 \right\} \tag{7}$$

Attention must be paid to the construction of the RID, in order to preserve coupling terms between the LFE and far field modes. Same considerations can be found in [2] for Navier-Stokes equations. This point is essential because the far field modes aim at propagating boundary conditions toward Ω_F.

Since $\Omega_F \subset \widehat{\Omega}$, the indices of $\mathcal{I}$ can be supported by nodes either on $\mathbf{\Gamma}$ or inside $\widehat{\Omega}$. This affect the coupling terms in the setting of momentum equations for PMFF.

Let's consider the Jacobian matrix of the FE approximation. It has the following block shape:

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{RR} & \mathbf{J}_{RF} \\ \mathbf{J}_{FR} & \mathbf{J}_{FF} \end{bmatrix} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}, \quad \mathbf{J}_{FF} \in \mathbb{R}^{\text{Card}(\mathcal{F})\times\text{Card}(\mathcal{F})} \tag{8}$$

such that:

$$j_{RF\,ip} = 0 \quad \forall\, i \notin \mathcal{I},\ i \leq \mathcal{N} - \text{Card}(\mathcal{F}),\ p \in \{1, \ldots, \text{Card}(\mathcal{F})\} \tag{9}$$

The hyper-reduced setting of the Newton Raphson's linear step reads: find $\delta\gamma \in \mathbb{R}^{N+\text{Card}(\mathcal{F})}$ such that

$$\mathbf{V}[\mathcal{L}, :]^T\, \mathbf{r}[\mathcal{L}] + \mathbf{V}[\mathcal{L}, :]^T\, \mathbf{J}[\mathcal{L}, :]\, \mathbf{V}\, \delta\gamma = 0 \tag{10}$$

Attention must be paid to the first row block in $\mathbf{J}$. In this first row block the contribution of the coupling terms reads:

$$\mathbf{V}[\mathcal{L}^R, :]^T\, \mathbf{J}_{RF}[\mathcal{L}^R, :]\, \mathbf{V}[\mathcal{F}, :] \tag{11}$$

We propose to include $\mathcal{I}$ to $\mathcal{H}$ after the selection of interpolation indices, during the RID construction, in order to have non zero terms in $\mathbf{J}_{RF}[\mathcal{L}^R, :]$. This preserves a strong coupling between le LFE approximation and the far field modes.

3 Numerical example

We consider a cantilever beam with perfect plasticity and a spherical pore in it, as shown in Figure 1. The far field modes have been computed without any pore. The error on the prediction of the cumulated plastic strain is shown on Figure 1, over the RID only. The maximum error

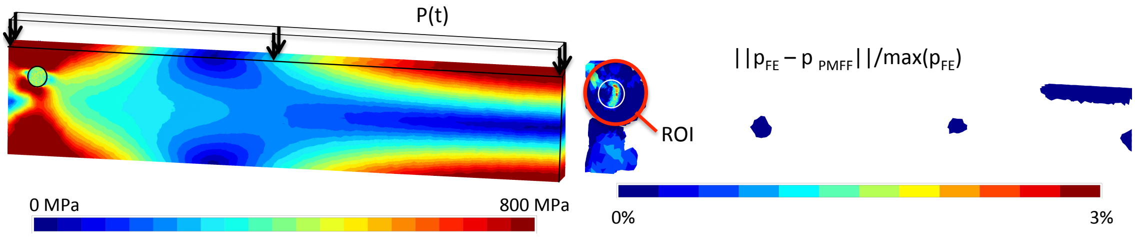

Figure 1. On the left, the von Mises stress predicted by the finite element method; on the right, the relative error on the cumulated plastic strain over the reduced integration domain related to the partial mechanics of far fields.

is 5%. The simulation speedup is 4. Obviously, the larger the pore's volume fraction the less accurate the prediction.

Plasticity under non parametric morphological modifications is accurately predicted by the hybrid-hyper-reduced order model. This open a new route to consider defects in materials such as pores or inclusions. It is termed the partial mechanics of far fields.

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