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Thermo-magneto-mechanical modeling of magneto-rheological elastomers

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Micro Abstract
A general thermodynamically consistent constitutive framework for thermo-magneto-mechanically coupled phenomena is devised in this contribution. A generalized formulation for the total thermo-magneto-mechanical energy function in an additive form is presented where the magneto-mechanically coupled effect is linearly scaled with the temperature. The framework is verified using a classical non-linear boundary value problem.

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Introduction

In the recent years a growing interest in the study of so-called smart materials in the finite deformations regime emerged. In this context especially magnetorheological elastomers (MREs) are a promising class of materials. MREs react with large deformations and a change in their mechanical properties in response to external excitations by a magnetic field which makes them interesting candidates for applications such as tunable stiffness and damping devices.

1 Basics of non-linear magneto-mechano-statics

1.1 Kinematics

Since polymeric materials typically can undergo large deformations we distinguish between the material configuration $\mathcal{B}_0$ and the spatial configuration $\mathcal{B}_t$. To describe the deformation of the body material coordinates $\boldsymbol{X}$ in $\mathcal{B}_0$ are mapped through the nonlinear deformation map $\boldsymbol{\chi}$ onto the spatial coordinates $\boldsymbol{x}$ in $\mathcal{B}_t$. The deformation gradient $\boldsymbol{F}$ is defined as the gradient of the deformation map $\boldsymbol{\chi}$ with respect to the material coordinates $\boldsymbol{X}$, i.e.

$$\boldsymbol{F} := \text{Grad}\,\boldsymbol{\chi}; \quad J := \det \boldsymbol{F} > 0, \tag{1}$$

where J is the Jacobian determinant of the deformation gradient that has to be positive in order to avoid any unphysical deformations.

1.2 Balance laws

1.2.1 Material configuration

Within a material body, the relation between the magnetic field $\mathbb{H}$ and the magnetic induction $\mathbb{B}$ is given in terms of the magnetization $\mathbb{M}$ and the magnetic permeability in vacuum μ_0

$$\mathbb{B} = J\mu_0 \boldsymbol{C}^{-1}[\mathbb{H} + \mathbb{M}] \quad \text{in } \mathcal{B}_0, \tag{2}$$

where $\boldsymbol{C}^{-1}$ is the inverse of the right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$. If we assume the magnetostatic case where the free current density is zero and the electric displacement is constant in time, Ampere's law together with the absence of magnetic monopoles yields

$$\text{Curl}\ \mathbb{H} = \boldsymbol{0}, \quad \text{Div}\ \mathbb{B} = 0, \tag{3}$$

where Curl and Div denote the corresponding differential operators with respect to the position vectors $\boldsymbol{X}$ in $\mathcal{B}_0$. On the boundary between the material and the free space the jump $[\![\bullet]\!]$ is defined as the difference of a quantity with regard to the outward pointing normal vector $\boldsymbol{n}$, i.e.: $[\![\bullet]\!] = \{\bullet\}^{\text{out}} - \{\bullet\}^{\text{in}}$. For the magnetic field and the induction this leads to

$$\boldsymbol{N} \cdot [\![\mathbb{B}]\!] = 0 \quad \text{and} \quad \boldsymbol{N} \times [\![\mathbb{H}]\!] = \hat{\mathbb{J}}^f \tag{4}$$

where $\hat{\mathbb{J}}^f$ denotes the free surface current density. As a stress measure in the material configuration we define the total Piola stress tensors $\boldsymbol{P}^{\text{tot}}$ as the combination of the mechanical Piola stress $\boldsymbol{P}$ and the ponderomotive Piola stress $\boldsymbol{P}^{\text{pon}}$

$$\boldsymbol{P}^{\text{tot}} = \boldsymbol{P} + \boldsymbol{P}^{\text{pon}} = \boldsymbol{P} + \boldsymbol{P}^{\text{mag}} + \boldsymbol{P}^{\text{max}}. \tag{5}$$

The ponderomotive contribution can be decomposed into the magnetization stress $\boldsymbol{P}^{\text{mag}}$ and the Maxwell stress $\boldsymbol{P}^{\text{max}}$. This term can be transformed into its spatial counterpart, the total Cauchy type stress $\boldsymbol{\sigma}^{\text{tot}}$ by the relation $\boldsymbol{\sigma}^{\text{tot}} = J^{-1}\boldsymbol{P}^{\text{tot}}\boldsymbol{F}^T$. The mechanical behavior is governed by the balance of linear momentum in combination with the mechanical tractions $\boldsymbol{t}_0^{\text{p}}$

$$\text{Div } \boldsymbol{P}^{\text{tot}} + \boldsymbol{b}_0 = \boldsymbol{0} \quad \text{with} \quad [\![\boldsymbol{P}^{\text{tot}}]\!] \cdot \boldsymbol{N} = -\boldsymbol{t}_0^p \quad \text{on } \partial\mathcal{B}_0^t. \tag{6}$$

2 Non-linear thermo-magneto-elasticity

2.1 Constitutive equations

We express the total energy function as $\Omega(\boldsymbol{F}, \Theta, \mathbb{H}) = \Psi(\boldsymbol{F}, \Theta, \mathbb{H}) + M_0^*(\boldsymbol{F}, \mathbb{H})$, with the free field magnetic complementary energy $M_0^*(\boldsymbol{F}, \mathbb{H})$ per unit undeformed volume. In the absence of a free current density, the Clausius-Duhem inequality can be expressed as

$$\delta_0 = \boldsymbol{P}^{\text{tot}} : \dot{\boldsymbol{F}} - \mathbb{B} \cdot \dot{\mathbb{H}} - \dot{\Omega} - H\dot{\Theta} - \boldsymbol{Q} \cdot \frac{\text{Grad}(\Theta)}{\Theta} \geq 0, \tag{7}$$

where H is the entropy and $\boldsymbol{Q}$ is the heat flux vector in the material configuration. Now, we can express the constitutive relations in terms of the total energy as

$$\boldsymbol{P}^{\text{tot}} = \frac{\partial \Omega}{\partial \boldsymbol{F}}, \quad \text{with} \quad \boldsymbol{P}^{\text{max}} = \frac{\partial M_0^*}{\partial \boldsymbol{F}}, \quad \mathbb{B} = -\frac{\partial \Omega}{\partial \mathbb{H}}, \quad H = -\frac{\partial \Omega}{\partial \Theta}, \tag{8}$$

see, e.g. [3] for further details. After applying the Coleman-Noll argumentation to Equation (7), the reduced conductive dissipation power density reads

$$\delta_0^{\text{con}} = -\boldsymbol{Q} \cdot \frac{\text{Grad}(\Theta)}{\Theta} \geq 0. \tag{9}$$

2.2 Energy function

In order to derive a thermo-magneto-mechanically coupled energy function we assume that the heat capacity at constant deformation and constant magnetic fields $c_{\boldsymbol{F},\mathbb{H}}$ is constant, i.e. $c_{\boldsymbol{F},\mathbb{H}}(\Theta) = c_{\boldsymbol{F},\mathbb{H}}(\Theta_0) = c_0$, whereby Θ_0 is the reference temperature. From the definition of c_0 we obtain

$$c_0 \quad = \quad -\Theta \frac{\partial^2 \Psi}{\partial \Theta \partial \Theta} \overset{!}{=} \text{const.} \Rightarrow -\frac{c_0}{\Theta} = \frac{\partial^2 \Psi}{\partial \Theta \partial \Theta}, \quad \text{with} \quad \Psi = \Psi(\boldsymbol{F}, \Theta, \mathbb{H}). \tag{10}$$

If the above relation is integrated twice from Θ_0 to an arbitrary temperature Θ, it becomes

$$\Psi = c_0 \Big[\Theta - \Theta_0 - \Theta \ln\Big(\frac{\Theta}{\Theta_0}\Big)\Big] - \Big[\Theta - \Theta_0\Big] M_1(\boldsymbol{F}, \mathbb{H}) + W(\boldsymbol{F}, \mathbb{H}). \tag{11}$$

where the integration constant M_1 may depend on the deformation gradient $\boldsymbol{F}$ and the magnetic field $\mathbb{H}$, which can be decomposed additively into a purely mechanical part $M(\boldsymbol{F})$ and a magneto-mechanically coupled part $C(\boldsymbol{F},\mathbb{H})$, i.e. $M_1(\boldsymbol{F},\mathbb{H}) = M(\boldsymbol{F}) + C(\boldsymbol{F},\mathbb{H})$. For isotropy, the isothermal energy function W (a function in $\boldsymbol{F}$ and $\mathbb{H}$) at the reference temperature depends on the magneto-mechanical coupled invariants, i.e. I_1 to I_6 as $W(\boldsymbol{F},\mathbb{H}) = W(I_1,\cdots I_6)$. As discussed in [2], in the case of large deformations, there are various forms to express the purely mechanical part $M(\boldsymbol{F})$. One of the simplest forms could be $M(\boldsymbol{F}) = 3\kappa\beta\ln(J)$, where κ is the bulk modulus coefficient at the reference temperature and β is the thermal expansion coefficient. For the magneto-mechanically coupled part $C(\boldsymbol{F},\mathbb{H})$ we assume a relation in line with the one proposed by Vertechy et al. [1] for thermo-electro-elasticity, which can be obtained by assuming

$$C(\boldsymbol{F},\mathbb{H}) = -\frac{1}{\Theta_0}W(\boldsymbol{F},\mathbb{H}). \tag{12}$$

This eventually yields a complete thermo-magneto-mechanically coupled energy function as

$$\Psi(\boldsymbol{F},\Theta,\mathbb{H}) = \frac{\Theta}{\Theta_0}W(\boldsymbol{F},\mathbb{H}) + c_0\Big[\Theta - \Theta_0 - \Theta\ln\Big(\frac{\Theta}{\Theta_0}\Big)\Big] - \Big[\Theta - \Theta_0\Big]M(\boldsymbol{F}). \tag{13}$$

To obtain a full expression of the temperature-dependent energy function derived in Equation (13), we need to define an isothermal energy function $W(\boldsymbol{F},\mathbb{H})$ at the reference temperature. For the sake of simplicity, a coupled incompressible Neo-Hookean-type material law depending on the invariants I_1, I_4 and I_5 is proposed. We assume that the shear modulus, due to its field-responsive nature depends on the applied magnetic field. For an increase in the stiffness due to magnetization and the phenomenon of magnetic saturation after a critical value of magnetization, a hyperbolic function such as $\mu_e/4\,[1+\alpha_e\tanh(I_4/m_e)]$ is assumed, where μ_e is the shear modulus of the material in the absence of a magnetic field

$$W(\boldsymbol{F},\mathbb{H}) = \frac{\mu_e}{4}\left[1 + \alpha_e\tanh\left(\frac{I_4}{m_e}\right)\right][I_1 - 3] + c_1 I_4 + c_2 I_5. \tag{14}$$

The parameter m_e is required for the purpose of non-dimensionalisation while α_e is a dimensionless positive parameter for scaling. The parameters c_1 and c_2 relate to the magneto-mechanical coupling. For $\alpha_e = c_1 = c_2 = 0$, this simplifies to the classical Neo-Hooke elastic energy density function widely used to model elastomers.

2.3 Magneto-mechanically coupled heat equation

From the first law of thermodynamics, the governing equation for the evolution of the thermal field can be written in entropy form as

$$\Theta\dot{H} = \mathcal{R} - \mathrm{Div}\boldsymbol{Q} + \mathcal{D}^{\mathrm{loc}} \quad \text{with } \mathcal{D}^{\mathrm{loc}} \equiv 0, \tag{15}$$

with the heat source $\mathcal{R}$ and the heat flux vector $\boldsymbol{Q}$ in the material configuration. By combining Equation (15) with the constitutive relation $H = -\frac{\partial\Psi}{\partial\Theta}$ we obtain the heat conduction equation in the format

$$c_0\dot{\Theta} = \mathcal{R} - \mathrm{Div}\boldsymbol{Q} + \Theta\partial_\Theta\Big[\boldsymbol{P}^{\mathrm{tot}} : \dot{\boldsymbol{F}} + \mathbb{B}\cdot\dot{\mathbb{H}}\Big]. \tag{16}$$

In contrast to the classical heat equation, this format contains two additional contributions. The structural thermo-mechanical cooling/heating effect related to $\dot{\boldsymbol{F}}$ and the thermo-magnetic heating/cooling effect related to $\dot{\mathbb{H}}$, see Vertechy et al. [1], Mehnert et al. [2] for a similar expression in the case of thermo-electro-elasticity.

3 Non-homogeneous boundary value problem

A cylindrical tube is deformed under a combination of axial extension, due to the normal force $\mathcal{N}$, and radial expansion that is the result of a pressure P on the internal surface of the tube. Simultaneously a radial temperature gradient and an azimuthal magnetic field are applied.

The geometry of the tube in the spatial (lower case letters) and in the material configuration (upper case letters) is described by

$$\begin{aligned} a_i \leq & r \leq a_e; \quad 0 \leq \phi \leq 2\pi; \quad 0 \leq z \leq l, \\ A_i \leq & R \leq A_e; \quad 0 \leq \Phi \leq 2\pi; \quad 0 \leq Z \leq L. \end{aligned} \tag{17}$$

In the considered case it is reasonable to work in the cylindrical coordinates (R, Φ, Z) with the unit basis vectors $(\boldsymbol{E}_R, \boldsymbol{E}_\Phi, \boldsymbol{E}_Z)$ defined in the material configuration. In the spatial configuration these quantities are defined as (r, ϕ, z) and $(\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z)$, respectively. Thus the transformation from the undeformed to the deformed configuration reads

$$r^2 = \lambda_z^{-1} \left[R^2 - A_i^2 \right] + a_i^2, \quad \phi = \Phi, \quad z = \lambda_z Z, \tag{18}$$

where the first relation is based on the incompressibility assumption and λ_z is the uniform axial stretch. This results in a deformation gradient that only has entries on the main diagonal. In cylindrical coordinates the radial, circumferential and axial entries read

$$\lambda_r = [\lambda \lambda_z]^{-1}; \quad \lambda_\phi = \frac{r}{R} = \lambda; \quad \lambda_z. \tag{19}$$

We assume that the outer surface of the tube is free of mechanical loads, i.e. $\sigma_{rr}^{\text{tot}}(a_e) = \sigma_{rr}^{\text{max}}(a_e)$, whereas the pressure P acts at the internal surface. This results in the total radial Cauchy stress

$$\sigma_{rr}^{\text{tot}}(a_i) = \sigma_{rr}^{\text{max}}(a_i) - P. \tag{20}$$

Using these boundary conditions in combination with the governing equations, the heat equation and the thermo-electro-mechanical framework an expression for both P and $\mathcal{N}$ can be derived giving an explicit characterization of the material behavior taking into account the influence of the applied non-mechanical fields.

Conclusions

In this contribution, we present a thermo-magneto-mechanically coupled framework for magneto-rheological elastomers that can operate in finite deformations. Departing from relevant laws of thermodynamics, we derive a thermodynamically consistent formulation in which temperature is an independent variable in addition to the mechanical and magnetic fields. In order to demonstrate the applicability of our proposed constitutive framework, a non-homogeneous boundary value problem that has frequently been used in finite elasticity and magneto-elasticity is presented. In this example a cylindrical thick-walled tube is subjected to a combination of radial inflation and axial extension, while simultaneously a circumferential magnetic field and a radial temperature gradient are applied.

References

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