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Hybrid Additive/Multiplicative Schwarz Preconditioning for Monolithic Solvers for Surface-Coupled Multi-Physics Problems

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Micro Abstract
We propose a hybrid additive/multiplicative Schwarz preconditioner for the monolithic solution of surface-coupled problems. Existing physics-based block preconditioners have proven to be very powerful but accumulate the error at the coupling surface. We address this issue by combining them with an additional additive Schwarz preconditioner, whose subdomains span across the interface on purpose. By performing cheap but accurate subdomain solves this error accumulation can be reduced efficiently.

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Introduction

Physical phenomena that include the coupling of multiple physical fields are omnipresent in our world. One prominent representative is the interaction of an incompressible fluid flow with solid bodies undergoing finite deformation, commonly referred to as fluid-structure interaction (FSI). Although already being studied for decades, solving such problems efficiently still poses a great challenge for numerical algorithms. Promising approaches are monolithic solvers as described by Mayr *et al.* [5] for example. They require powerful preconditioning techniques like physics-based block preconditioners utilizing algebraic multigrid (AMG) methods [3].

Here, we propose a novel *hybrid additive/multiplicative* Schwarz *preconditioner* based on an overlapping domain decomposition with subdomains intentionally spanning across the fluid-structure interface. To address the issue of error accumulation at the coupling surface in case of physics-based block preconditioners, we combine them with an additional additive Schwarz preconditioner whose subdomain solves are insensitive to the presence of the interface.

Fluid-Structure Interaction in a Nutshell

To establish a monolithic solution method for the coupled FSI problem, where all equations are solved simultaneously, spatial and temporal discretization are performed field-wise before the final assembly of the monolithic system of equations. For the spatial discretization of the solid and the fluid field we employ the finite element method.

Temporal discretization is done by finite differencing. For time integration, we use fully implicit, single-step, single-stage time integration schemes. In the solid field, we employ the generalized-α method [1]. The fluid field either uses the generalized-α method [4] or the one-step-θ scheme [2].

Putting the residual expressions $\mathbf{r}^{\mathcal{S}}$, $\mathbf{r}^{\mathcal{G}}$ and $\mathbf{r}^{\mathcal{F}}$ from the solid, the ALE, and the fluid field as well as the kinematic constraint $\mathbf{r}^{\mathrm{coupl}}$ together yields the monolithic nonlinear residual vector ${\mathbf{r}^{\mathrm{FSI}}}^{\mathrm{T}} = \left[\ \mathbf{r}^{\mathcal{S}}\ \mathbf{r}^{\mathcal{G}}\ \mathbf{r}^{\mathcal{F}}\ \mathbf{r}^{\mathrm{coupl}}\ \right]^{\mathrm{T}}$ that needs to vanish in every time step. The nonlinearity is treated by a Newton–Krylov method with FSI-specific preconditioning. After assembly, consistent linearization, and subsequent static condensation of the LAGRANGE multiplier and slave side interface degrees of freedom, the monolithic system of linear equations schematically reads

$$\begin{bmatrix} \mathbf{\mathcal{S}} & & \mathbf{\mathcal{C}}^{\mathcal{S}\mathcal{F}} \\ & \mathbf{\mathcal{A}} & \mathbf{\mathcal{C}}^{\mathcal{G}\mathcal{F}} \\ \mathbf{\mathcal{C}}^{\mathcal{F}\mathcal{S}} & \mathbf{\mathcal{C}}^{\mathcal{F}\mathcal{G}} & \mathbf{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}^{\mathcal{S}} \\ \Delta\mathbf{x}^{\mathcal{G}} \\ \Delta\mathbf{x}^{\mathcal{F}} \end{bmatrix} = - \begin{bmatrix} \mathbf{r}^{\mathcal{S}} \\ \mathbf{r}^{\mathcal{G}} \\ \mathbf{r}^{\mathcal{F}} \end{bmatrix}. \tag{1}$$

The matrices $\mathbf{\mathcal{S}}$, $\mathbf{\mathcal{A}}$, and $\mathbf{\mathcal{F}}$ on the main diagonal reflect the solid, the ALE, and the fluid field residual linearizations, respectively. The coupling among the fields is represented by the off-diagonal blocks $\mathbf{\mathcal{C}}^{ij}$, where superscripts $i, j \in \{\mathcal{S}, \mathcal{G}, \mathcal{F}\}$ indicate the coupling between the fields.

The Hybrid Preconditioner

To set up a combined preconditioner two building blocks are necessary, namely one physics-based block preconditioner $\mathbf{\mathcal{M}}_{\text{MS}}^{-1}$ plus the additional preconditioner $\mathbf{\mathcal{M}}_{\text{AS}}^{-1}$ based on the partition of the domain in subdomains.

For $\mathbf{\mathcal{M}}_{\text{AS}}^{-1}$ the block structure of the system matrix is of no importance. Sorting all unknowns by their affiliation to subdomains yields the matrix representation

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & \cdots & \mathbf{A}_{0n} \\ \mathbf{A}_{10} & \mathbf{A}_{11} & \cdots & \mathbf{A}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n0} & \mathbf{A}_{n1} & \cdots & \mathbf{A}_{nn} \end{bmatrix} \tag{2}$$

distributed among n subdomains, where n usually equals the number of processes n^{proc}. Matrices $\mathbf{A}_{ii}$ are restrictions of the global matrix $\mathbf{A}$ to process i, while off-diagonal matrices $\mathbf{A}_{ij}$ and $\mathbf{A}_{ji}$ account for the coupling between the local subproblems on processes i and j. All process-local matrices in (2), especially the off-diagonal ones, are sparse. The additional preconditioner $\mathbf{\mathcal{M}}_{\text{AS}}^{-1}$ is obtained by dropping all off-diagonal coupling blocks in (2), which results in the additive SCHWARZ preconditioner

$$\mathbf{\mathcal{M}}_{\text{AS}}^{-1} = \begin{bmatrix} \mathbf{A}_{00} & & & \\ & \mathbf{A}_{11} & & \\ & & \ddots & \\ & & & \mathbf{A}_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{00}^{-1} & & & \\ & \mathbf{A}_{11}^{-1} & & \\ & & \ddots & \\ & & & \mathbf{A}_{nn}^{-1} \end{bmatrix}.$$

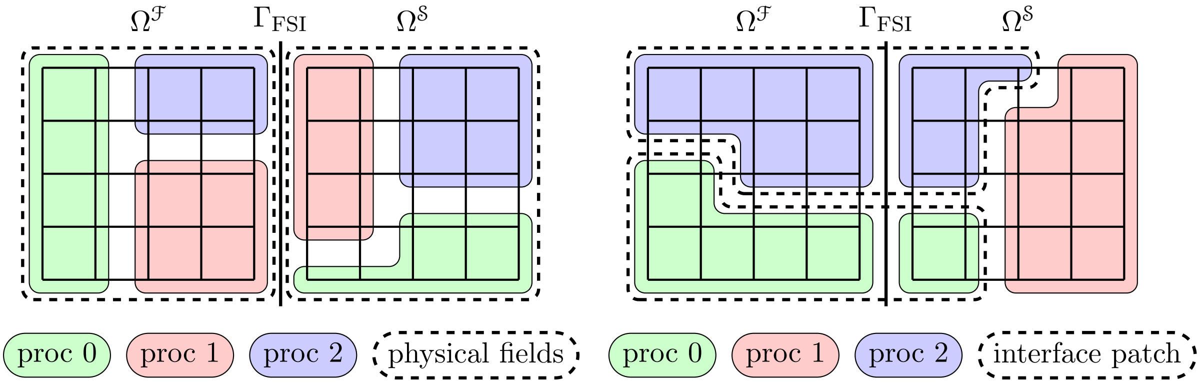
In order to tackle error accumulation due to physics-based block preconditioners the subdomains must span across the interface, see Figure Figure 1.

Physics-based block preconditioners $\mathbf{\mathcal{M}}_{\text{MS}}^{-1}$ like *block* GAUSS–SEIDEL *(BGS)* are formally of multiplicative SCHWARZ type, which has been indicated by the subscript MS. Respectively, the notation $\mathbf{\mathcal{M}}_{\text{AS}}^{-1}$ of the additional preconditioner with its subscript AS refers to additive SCHWARZ methods.

The physics-based block preconditioner $\mathbf{\mathcal{M}}_{\text{MS}}^{-1}$ and the additional additive SCHWARZ preconditioner $\mathbf{\mathcal{M}}_{\text{AS}}^{-1}$ are chained together to form the *hybrid additive/multiplicative* SCHWARZ *preconditioner*. It is applied in a multiplicative SCHWARZ fashion, reading

$$\mathbf{\mathcal{M}}_{\text{HS}}^{-1} = \mathbf{\mathcal{M}}_{\text{AS}}^{-1} \circ \mathbf{\mathcal{M}}_{\text{MS}}^{-1} \circ \mathbf{\mathcal{M}}_{\text{AS}}^{-1} \tag{3}$$

where the additive SCHWARZ preconditioner is applied before and after the physics-based block preconditioner. In GMRES iteration k, the preconditioner (3) is applied to the linear system via



(a) Physics-based overlapping domain decomposition

(b) Overlapping domain decomposition based on monolithic graph

Figure 1. Overlapping domain decompositions of a FSI problem — *Left:* At the fluid-structure interface Γ_{FSI}, the domain is partitioned into solid and fluid subdomains indicated by dashed lines. Each field can further be distributed among several processes by an overlapping domain decomposition indicated by the colored patches. Overlap of subdomains is not depicted for clarity of presentation. *Right:* The subdomains span across the interface like 'proc 0' and 'proc 2'. They are crucial for the effectiveness of the proposed preconditioner. Some processes might not own portions of both fields, e.g. 'proc 1'. Overlap of subdomains is not depicted for clarity of presentation.

three stationary RICHARDSON iterations

$$\begin{aligned}
\mathbf{x}_{\mathrm{I}}^{k} &= \mathbf{x}_{0}^{k} + \omega_{\mathrm{AS}} \mathcal{M}_{\mathrm{AS}}^{-1} \left(\mathbf{b} - \mathbf{A} \mathbf{x}_{0}^{k} \right) \\
\mathbf{x}_{\mathrm{II}}^{k} &= \mathbf{x}_{\mathrm{I}}^{k} + \omega_{\mathrm{MS}} \mathcal{M}_{\mathrm{MS}}^{-1} \left(\mathbf{b} - \mathbf{A} \mathbf{x}_{\mathrm{I}}^{k} \right) \\
\mathbf{x}_{\mathrm{III}}^{k} &= \mathbf{x}_{\mathrm{II}}^{k} + \omega_{\mathrm{AS}} \mathcal{M}_{\mathrm{AS}}^{-1} \left(\mathbf{b} - \mathbf{A} \mathbf{x}_{\mathrm{II}}^{k} \right)
\end{aligned} \tag{4}$$

with damping parameters ω_{AS} and ω_{MS} and the initial solution $\mathbf{x}_{0}^{k}$. Intermediate steps after the first and second RICHARDSON iteration are denoted by $\mathbf{x}_{\mathrm{I}}^{k}$ and $\mathbf{x}_{\mathrm{II}}^{k}$, respectively, while the final result of the preconditioning operation is referred to as $\mathbf{x}_{\mathrm{III}}^{k}$.

Numerical Experiments and Results

We study the performance of the proposed preconditioner using the pressure wave benchmark test, cf. [3, 5] among others. We compare our new approach to two powerful variants of FSI-specific multigrid preconditioners, namely an outer block GAUSS–SEIDEL (BGS) method with AMG-based approximations of block inverses and an AMG hierarchy of the coupled problem with BGS level smoothers [3]. These physics-based approaches are referred to as BGS(AMG) and AMG(BGS), respectively. Solid and ALE blocks are treated with Smoothed Aggregation AMG with Chebychev polynomials as level smoothers, while the fluid block is addressed with PETROV–GALERKIN AMG with damped symmetric GAUSS–SEIDEL level smoothers. Direct solvers are used on the coarse level in every field. For the hybrid preconditioner (3), we choose $\mathcal{M}_{\mathrm{MS}}^{-1}$ to be BGS(AMG) or AMG(BGS) and use ILU(0) factorizations for the subdomain solves in $\mathcal{M}_{\mathrm{AS}}^{-1}$. Depending on the choice for $\mathcal{M}_{\mathrm{MS}}^{-1}$, we denote the hybrid variants by H-BGS(AMG) and H-AMG(BGS), respectively. GMRES iteration counts as well as pure solver time and solver time including preconditioner setup are reported in Figure 2 for a problem with 1,948,161 unknowns solved on 256 cores. Iteration counts and pure solver time are reduced by at least 20% by the hybrid preconditioner, while total solver time including setup of the preconditioner is reduced by at least 15%. In addition to these remarkable computational savings, weak scalability of the proposed preconditioner could be demonstrated.

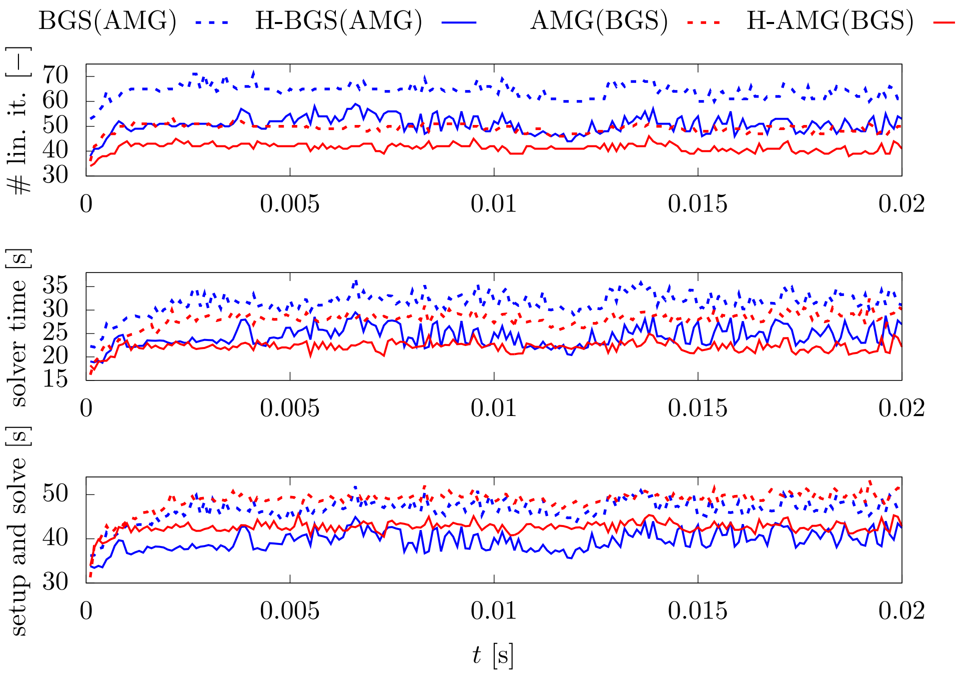


Figure 2. The comparison of GMRES iteration counts and timings of purely physics-based block preconditioners and their hybrid counterparts demonstrates the efficiency of the novel preconditioner.

Concluding Remarks

We proposed a novel hybrid preconditioner for monolithic FSI solvers, that combines powerful physics-based block preconditioners with an additional additive SCHWARZ preconditioner whose subdomain solves are insensitive to the presence of the interface. The novel approach reduces computational cost by at least 20% in large-scale examples and additionally exhibits weak scalability.

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