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A posteriori error estimates for finite elements of higher-order for frictional, elasto-plastic two-body contact problem

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Micro Abstract
We present a residual a posteriori error estimator for frictional, elasto-plastic two-body contact problems and finite elements of higher order. It is based on a mixed formulation in which the constraints concerning contact, friction and plasticity are captured by Lagrange multipliers. To be able to apply a semi-smooth Newton method we solve a primal-mixed problem and calculate the plastic quantities in a post process. Reliability and suboptimal efficiency of the estimator are shown.

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Introduction

Elasto-plastic two-body contact problems play an important role in the simulation of many manufacturing processes for instance in metal forming. Thus, there is a high interest in efficient and accurate discretizations as well as solving algorithms. In this article, we focus on the discretization aspect and shortly discuss a residual type a posteriori error estimator for this problem class using a higher-order finite element discretization and a mesh adaptive algorithm based on it. We obtain reliability of the estimator but only suboptimal efficiency. The adaptive algorithm is applied on a 2d example, in which it shows optimal convergence rates for different polynomial degrees.

1 Strong and weak problem formulation

We consider two contacting deformable bodies $\Omega^m \subset \mathbb{R}^d$ with $m = 1, 2$ and $d = 2, 3$ using a elasto-plastic material law with linear isotropic hardening. Here, volume forces f^m act on them. The boundaries are given by Γ^m, $m = 1, 2$. With the outer normal vector n, we define $\sigma_n(u^m, p^m) := \sigma(u^m, p^m)n$, $\sigma_{nn} := n^\top \sigma(u^m, p^m)n$, and $\sigma_{nt,i} := n^\top \sigma(u^m, p^m)t_i$ with corresponding tangential vectors t_i, $i = 1, \dots, d-1$. We are interested in the displacements $u = (u^1, u^2)$, which fulfill for $m = 1, 2$ the following equations:

$$\varepsilon(u^m) = A^m\sigma(u^m, p^m) + p^m \quad \text{in } \Omega^m \tag{1}$$

$$-\operatorname{div}\sigma(u^m, p^m) = f^m \quad \text{in } \Omega^m \tag{2}$$

$$p^m(\tau - \sigma(u^m, p^m)) \geq 0 \quad \forall\tau \text{ with } \mathcal{F}^{m,\mathrm{iso}}(\tau, |p^m|) \leq 0 \quad \text{in } \Omega^m \tag{3}$$

$$u^m = 0 \quad \text{on } \Gamma_D^m \tag{4}$$

$$\sigma_n(u^m, p^m) = f_N^m \quad \text{on } \Gamma_N^m\ . \tag{5}$$

Equation (1) describes the relation between the linearized strains $\varepsilon(u^m)$ and the stresses $\sigma(u^m, p^m)$. The strains are additively split in an elastic part $A^m\sigma(u^m, p^m)$ and a plastic part p^m. The deviatoric part of a tensor τ is denoted by $\operatorname{dev}(\tau)$. The flow function is defined by $\mathcal{F}^{m,\mathrm{iso}}(\tau, \eta) = |\operatorname{dev}(\tau)| - (\sigma_0^m + H^m\eta)$ with the yield stress σ_0^m, the isotropic hardening parameter H^m, and the equivalent plastic stress η. Both bodies are fixed on a nonempty set $\Gamma_D^m \subset \Gamma^m$. On the Neumann boundary Γ_N^m, surface forces f_N^m are applied.

On the subsets $\Gamma_C^m \subset \Gamma^m$, contact between the two bodies can occur. Let $\Phi : \Gamma_C^1 \to \Gamma_C^2$ be a bijective and smooth map between the contact boundary of the slave Ω^1 and on the master Ω^2. Furthermore, we denote by n_δ a generalized normal vector and corresponding tangential vectors t_δ. On Γ_C^1 the jump in direction of the generalized normal is given by $[v]_{n_\delta}$ and the one in tangential direction by $[v]_{t_\delta}$. The distance in the initial configuration is denoted by g. All in all, we obtain the following geometrical contact conditions on Γ_C^1:

$$[u]_{n_\delta} \leq g, \quad \sigma_{n_\delta n_\delta}(u^1) \leq 0, \quad \sigma_{n_\delta n_\delta}(u^1)([u]_{n_\delta} - g) = 0 \tag{6}$$

$$\sigma_{n_\delta}(u^1) = -\Theta^* \sigma_{n_\delta}(u^2). \tag{7}$$

Here, (7) ensures the equality of the contact stresses on both contact boundaries. Furthermore, we consider frictional side conditions on Γ_C^1:

$$\left|\sigma_{n_\delta t_\delta}(u^1)\right| \leq s(\sigma_{n_\delta n_\delta}(u^1)) \tag{8}$$

$$\left|\sigma_{n_\delta t_\delta}(u^1)\right| < s(\sigma_{n_\delta n_\delta}(u^1)) \Rightarrow [u]_{t_\delta} = 0 \tag{9}$$

$$\left|\sigma_{n_\delta t_\delta}(u^1)\right| = s(\sigma_{n_\delta n_\delta}(u^1)) \Rightarrow \exists \alpha \in \mathbb{R}_{\geq 0} : [u]_{t_\delta} = \alpha \sigma_{n_\delta t_\delta}(u) \tag{10}$$

The tangential stress $\sigma_{n_\delta t_\delta}(u^1)$ ist bounded by the frictional resistance s representing a general friction law depending only on the normal contact stress.

Now, we give the weak problem formulation. It is based on the following function spaces: $V := H_D^1(\Omega^1, \mathbb{R}^d) \times H_D^1(\Omega^2, \mathbb{R}^d)$, $Q^m := \left\{ q \in L^2(\Omega^m; \mathbb{R}^{d\times d}_{\text{sym}}) |\, \text{tr}(q) = 0 \text{ a.e. in } \Omega^m \right\}$, $Q := Q^1 \times Q^2$, $W = V \times Q$, $\Lambda_n := \left\{ \mu \in \tilde{H}^{-\frac{1}{2}}(\Gamma_C^1) \;\middle|\; \forall v \in H_+^{1/2}(\Gamma_C^1) : \langle \mu, v \rangle \geq 0 \text{ a.e.} \right\}$,

$$\Lambda_t := \left\{ \mu \in \left(\tilde{H}^{-1/2}(\Gamma_C^1)\right)^{d-1} \;\middle|\; \langle \mu, [v]_t \rangle \leq \langle s(\lambda_n), [v]_t \rangle \text{ a.e. } \forall [v]_t \in \left(H^{1/2}(\Gamma_C^1)\right)^{d-1} \right\},$$

and $\Lambda_P := \{ \mu \in Q \mid \mu : \mu \leq 1 \}$. Furthermore, we define the following bilinear and linear forms:

$$\begin{aligned}
a : W \times W \to \mathbb{R}, \qquad & a(w, z) := \sum_{m=1}^{2} \left[(\sigma(u^m, p^m), \varepsilon(v^m) - q^m)_0 + (H^m p^m, q^m)_0 \right], \\
b_n : \Lambda_n \times W \to \mathbb{R}, \qquad & b_n(\mu, z) := \langle \mu, [v]_{n_\delta} \rangle, \\
b_t : \Lambda_t \times W \to \mathbb{R}, \qquad & b_t(\mu, z) := \langle \mu, [v]_{t_\delta} \rangle, \\
b_P : \Lambda_P \times W \to \mathbb{R}, \qquad & b_P(\mu, z) := \sum_{m=1}^{2} b_P^m(\mu^m, z^m) := \sum_{m=1}^{2} \left(\mu^m, \sigma_y^m q^m \right)_0, \\
\mathcal{F} : V \to \mathbb{R}, \qquad & \mathcal{F}(v) := \sum_{m=1}^{2} (f^m, v^m)_0 + (f_N^m, v^m)_{0,\Gamma_N^m}.
\end{aligned}$$

The weak problem consists in finding $(w, \lambda_P, \lambda_n, \lambda_t) \in W \times \Lambda_P \times \Lambda_n \times \Lambda_t$ with $w = (u, p)$ such that

$$a(w, z) + b_P(\lambda_P, z) + b_n(\lambda_n, z) + b_t(\lambda_t, z) = \mathcal{F}(z) \quad \forall z \in W \tag{11}$$

$$b_P(\mu_P - \lambda_P, w) + b_n(\mu_n - \lambda_n, w) + b_t(\mu_t - \lambda_t, w) \leq \langle \mu_n - \lambda_n, g \rangle \quad \forall\, (\mu_P, \mu_n, \mu_t) \in \Lambda_P \times \Lambda_n \times \Lambda_t. \tag{12}$$

2 Discretization

In this section, we introduce the discretization of the mixed problem (11)-(12). Let $\mathcal{T}_h^m$ be an admissible triangulation of Ω^m with mesh width $h > 0$ using quads or hexahedrons. The triangulation of the contact boundary Γ_C^1 is given by $\mathcal{B}_H$. We use the affine-linear transformations F_T^m and F_E as well as the vector space S_l^r of polynomials of order r on the reference element $[-1,1]^l$. We define the following discrete function spaces:

$$\begin{aligned}
\mathcal{M}_h^p(\Omega^m) &:= \left\{ v \in L^2(\Omega^m) \;\middle|\; \forall T \in \mathcal{T}_h^m : v|_T \circ\ F_T^m \in S_d^p \right\}, \\
V_{h,p_m}^m &:= \{ v \in H_D(\Omega_m, \mathbb{R}^d) \,|\, v \in \mathcal{M}_h^{p_m}(\Omega^m) \}, \quad V_h \;:=\; V_{h,p_1}^1 \times V_{h,p_2}^2, \\
Q_h &:= \left\{ q = \left(q^1, q^2\right) \in Q \,\middle|\, q_{ij}^m \in \mathcal{M}_h^p(\Omega^m),\, m = 1,2 \right\}, \quad W_h \;:=\; V_h \times Q_h, \\
\mathcal{M}_H^q &:= \left\{ v \in L^2(\Gamma_C^1) \;\middle|\; \forall E \in \mathcal{B}_H : v|_E \circ\ F_E \in S_{d-1}^q \right\}, \\
\Lambda_{n,H} &:= \left\{ v \in \mathcal{M}_H^q \;\middle|\; \forall E \in \mathcal{B}_H : \forall x \in \mathcal{C}_q : v(F_E(x)) \geq 0 \;\right\}, \\
\Lambda_{t,H} &:= \{ v \in (\mathcal{M}_H^q)^{d-1} \,|\, \forall E \in \mathcal{B}_H : \forall x \in \mathcal{C}_q : v(F_E(x)) \leq s(\lambda_n)(F_E(x)) \}, \\
\Lambda_{P,h} &:= \left\{ q \in Q_h \;\middle|\; \forall m \in \{1,2\} : \forall x \in \mathcal{G}^{m,d} : q(x) : q(x) \leq 1 \right\}.
\end{aligned}$$

The side conditions of the Lagrange multipliers are only defined on the finite set $\mathcal{C}_q \subset [-1,1]^{d-1}$, which consists in $(q+1)^{d-1}$ Gauß-points. Furthermore, let $\mathcal{G}^{m,d}(T)$ be the set of the transformed Gauß-points on T and $\mathcal{G}^{m,d} := \bigcup_{T \in \mathcal{T}_h^m} \mathcal{G}^{m,d}(T)$.

The discrete formulation of the mixed problem (11)-(12) reads: Find $(w_h, \lambda_{P,h}, \lambda_{n,H}, \lambda_{t,H}) \in W_h \times \Lambda_{P,h} \times \Lambda_{n,H} \times \Lambda_{t,H}$, such that

$$a(w_h, z_h) + \sum_{m=1}^{2} (\lambda_{P,h}^m, \sigma_0^m q_h^m)_0 + b_n(\lambda_{n,H}, z_h) + b_t(\lambda_{t,H}, z_h) = \mathcal{F}(z_h) \quad \forall z_h \in W_h \tag{13}$$

$$\begin{aligned} b_P(\mu_{P,h}^m - \lambda_{P,h}^m, w_h)_0 + b_n(\mu_{n,H} - \lambda_{n,H}, w_h) + b_t(\mu_{t,H} - \lambda_{t,H}, w_h) \leq \langle \mu_{n,H} - \lambda_{n,H}, g \rangle \\ \forall\ (\mu_{P,h}, \mu_{n,H}, \mu_{t,H}) \in \Lambda_{P,h} \times \Lambda_{n,H} \times \Lambda_{t,H}. \end{aligned} \tag{14}$$

We solve the discrete problem using the techniques described in [1], where the elasto-plastic part is formulated in primal form using a suitable projection. The plastic Lagrange multiplier $\lambda_{P,h}$ is determined in a simple post processing step, cf. [2, Section 4.3]. The discretization is inf-sup-stable provided the term $\sum_{m=1}^2 \left(hH^{-1} \max\{1,q\}^2 p^{-1} \right)$ is small enough indepently of the discretization parameters. Numerical experiments show that the choice $q = p - 1$ and $H = 2\max\{h^1, h^2\}$ leads to a stable discretization. However, this condition has to be ensured on adaptive meshes, locally.

Residual a posteriori error estimation

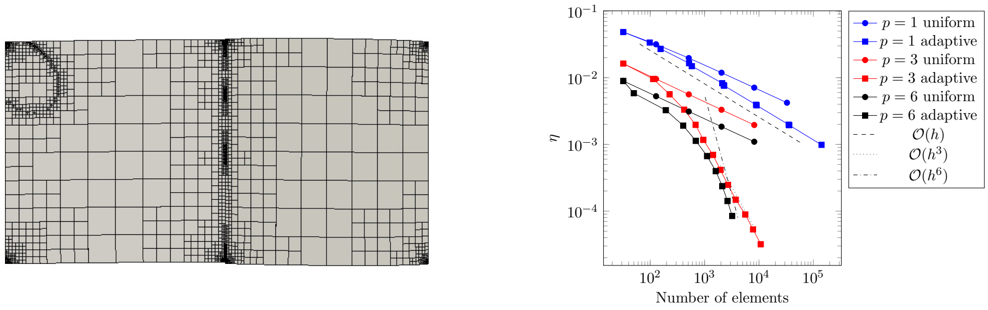
With respect to the reliability and efficiency of a residual type a posteriori error estimator, the following result holds:

Proposition 1. *There exist positve constants C, C_0, and C_1 such that*

$$\begin{aligned} &\|w - w_h\|_W^2 + \|\lambda_P - \lambda_{P,h}\|_0^2 + \|\lambda_n - \lambda_{n,H}\|_{-1/2}^2 + \|\lambda_t - \lambda_{t,H}\|_{-1/2}^2 \\ &\quad \leq C\eta^2\left((\lambda_{P,h})_1, (\lambda_{n,H})_+, (\lambda_{t,H})_s\right) \end{aligned}$$

and

$$\begin{aligned} \eta^2\left((\lambda_{P,h})_1, (\lambda_{n,H})_+, (\lambda_{t,H})_s\right) \leq C_0 \Bigg(&\|w - w_h\|_W^2 + \sum_{m=1}^{2} \|\lambda_P^m - \lambda_{P,h}^m\|_{0,\Gamma_C^1}^2 + \|\lambda_n - \lambda_{n,H}\|_{-1/2,\Gamma_C^1}^2 \\ &+ \|\lambda_t - \lambda_{t,H}\|_{-1/2,\Gamma_C^1}^2 \Bigg) + C_1 \|w - w_h\|_W + \mathrm{osc}_h^2(f_\Omega) + \mathrm{osc}_h^2(f_N). \end{aligned}$$



(a) Adaptive mesh (b) Convergence

Figure 1. Adaptive mesh for polynomial degree $p = 6$ and convergence results for different polynomial degrees.

Here, $(\lambda_{P,h})_1$, $(\lambda_{n,H})_+$, *and* $(\lambda_{t,H})_s$ *are projections on the admissible sets of the discrete Lagrange multipliers. The data oscillation concerning the given problem data is denoted by* osc_h. *The error estimator is given by*

$$\begin{aligned}\eta^2(\mu_P,\mu_n,\mu_t) := \eta_R^2 + \sum_{m=1}^{2} \|\lambda_{P,h}^m - \mu_P^m\|_0^2 + \|\lambda_{n,H} - \mu_n\|^2_{-1/2,\Gamma_C^1} + \|\lambda_{t,H} - \mu_t\|^2_{-1/2,\Gamma_C^1} \\ + \|([u_h]_n - g)_+\|^2_{1/2,\Gamma_C^1} + |(\lambda_{n,H}, ([u_h]_n - g)_+)_{0,\Gamma_C^1}| + |\langle \mu_n, g - [u_h]_n\rangle| \\ + \Psi_F(w_h) - \langle \mu_t, [u_h]_t\rangle + \Psi_P(w_h) - \sum_{m=1}^{2} (\mu_P^m, \sigma_0^m p_h^m)_0 \,,\end{aligned}$$

where η_R *is the standard residual error estimator,* Ψ_F *corresponds to the frictional energy and* Ψ_P *to the plastic one.*

Proof. [2, Korollar 1 and Satz 12]. □

Numerical results

We shortly present numerical results taken from [2, Beispiel 4]. Here, we consider an elastic-plastic two body contact problem in 2D with a complex nonlinear friction law. In Figure 1(a), an adaptive mesh for a polynomial degree of 6 is shown. Here, the boundary of the plastic zone and the switching points between stick and slip are well resolved. The behaviour of the adaptive algorithm is presented in Figure 1(b). We found here that the optimal order of convergence is obtained using the adaptive algorithm for the different polynomial degrees.

Conclusions

In this article, we have presented a residual a posteriori error estimator for elasto-plastic two body contact problems, which is reliable and suboptimal efficient. Numerical results substantiate that the adpative algorithm based on it leads to optimal convergence of the underlying discretization.

References

[1] H. Blum, H. Frohne, J. Frohne, and A. Rademacher. Semi-smooth newton methods for mixed FEM discretizations of higher-order for frictional, elasto-plastic two-body contact problems. *Computer methods in applied mechanics and engineering*, 309:p. 131–151, 2016.

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